Homotopy groups used in physics. For PHY 503.

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APPENDIX A: HOMOTOPY GROUPS

1. Generalities

If M and N are two topological spaces then for their direct product we have

$$\pi_k(M \times N) = \pi_k(M) \times \pi_k(N).$$

If M is a simply-connected topological space ($\pi_0(M) = \pi_1(M) = 0$) and group H acts on M then one can form topological space M/H identifying points of M which can be related by some element of H ($x \equiv hx$). We have

$$\pi_1(M/H) = \pi_0(H).$$

In particular, if H is a discrete group $\pi_0(H) = H$ and

$$\pi_1(M/H) = H.$$

For higher homotopy groups we have

$$\pi_k(M/H) = \pi_k(M), \quad \text{if } \pi_k(H) = \pi_{k-1}(H) = 0.$$

2. Homotopy groups of spheres

For a circle

$$\begin{aligned} \pi_1(S^1) &= Z, \\ \pi_k(S^1) &= 0, \quad \text{for } k \geq 2. \end{aligned}$$

For higher-dimensional spheres it is true that

$$\begin{aligned} \pi_n(S^n) &= Z, \\ \pi_k(S^n) &= 0, \quad \text{for } k < n. \end{aligned}$$

Homotopy groups of spheres $\pi_{n+k}(S^n)$ do not depend on n for $n > k+1$ (homotopy groups stabilize). In the table below we shade the cell from which homotopy groups remain constant (along the diagonal).

Homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
S^1	Z	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Here and thereon we denote Z the group isomorphic to the group of integer numbers with respect to an addition. Z_n is a finite Abelian cyclic group. It can be thought of as a group of n-th roots of unity with respect to a multiplication. Alternatively, it is isomorphic to a group of numbers $\{0, 1, 2, \ldots, n-1\}$ with respect to an addition modulo n. Or simply $Z_n = Z/nZ$.

3. Homotopy groups of Lie groups

a. Unitary groups

Bott periodicity theorem for unitary groups: for $k > 1$, $n \geq \frac{k+1}{2}$

$$\pi_k(U(n)) = \pi_k(SU(n)) = \begin{cases} 0, & \text{if } k\text{-even;} \\ Z, & \text{if } k\text{-odd.} \end{cases}$$

The fundamental group $\pi_1(SU(n)) = 0$ and $\pi_1(U(n)) = 1$ for all n.
In the following table we shade the cells from which Bott periodicity theorem "starts working".

Homotopy groups of unitary groups

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$U(1)$	Z	0	0	0	0	0	0	0	0	0	0	0
$U(2)$	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
$U(3)$	0	0	Z	0	Z	Z_6						
$U(4)$	0	0	Z	0	Z	0	Z					
$U(5)$	0	0	Z	0	Z	0	Z	0	Z			

b. Orthogonal groups

Bott periodicity theorem for orthogonal groups: for $n \geq k + 2$

$$\pi_k(O(n)) = \pi_k(SO(n)) = \begin{cases} 0, & \text{if } k = 2, 4, 5, 6 \pmod 8; \\ Z_2, & \text{if } k = 0, 1 \pmod 8; \\ Z, & \text{if } k = 3, 7 \pmod 8. \end{cases}$$

In the following table we shade the cells from which Bott periodicity theorem "starts working".

Homotopy groups of orthogonal groups

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$SO(2)$	Z	0	0	0	0	0	0	0	0	0	0	0
$SO(3)$	Z_2	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$(Z_2)^{\times 2}$
$SO(4)$	Z_2	0	$(Z)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_{12})^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_3)^{\times 2}$	$(Z_{15})^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 4}$
$SO(5)$	Z_2	0	Z	Z_2	Z_2	0	Z	0	0	Z_{120}	Z_2	$(Z_2)^{\times 2}$
$SO(6)$	Z_2	0	Z	0	Z	0	Z	Z_{24}	Z_2	$Z_{120} \times Z_2$	Z_4	Z_{60}
$SO(n), \ n > 6$	Z_2	0	Z	0	0	0						

c. Symplectic groups

Bott periodicity theorem for symplectic groups: for $n \geq \frac{k-1}{4}$

$$\pi_k(Sp(n)) = \begin{cases} 0, & \text{if } k = 0, 1, 2, 6 \pmod 8; \\ Z_2, & \text{if } k = 4, 5 \pmod 8; \\ Z, & \text{if } k = 3, 7 \pmod 8. \end{cases}$$

In the following table we shade the cells from which Bott periodicity theorem "starts working".

Homotopy groups of symplectic groups												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$Sp(1)$	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
$Sp(2)$	0	0	Z	Z_2	Z_2	0	Z	0	0	Z_{120}	Z_2	$Z_2 \times Z_2$
$Sp(3)$	0	0	Z	Z_2	Z_2	0	Z	0	0	0	Z	Z_2
$Sp(4)$	0	0	Z	Z_2	Z_2	0	Z	0	0	0	Z	Z_2
$Sp(5)$	0	0	Z	Z_2	Z_2	0	Z	0	0	0	Z	Z_2

d. Exceptional groups

Homotopy groups of exceptional groups												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
G_2	0	0	Z	0	0	Z_3	0	Z_2	Z_6	0	$Z \times Z_2$	0
F_4	0	0	Z	0	0	0	0	Z_2	Z_2	0	$Z \times Z_2$	0
E_6	0	0	Z	0	0	0	0	0	Z	0	Z	Z_{12}
E_7	0	0	Z	0	0	0	0	0	0	0	Z	Z_2
E_8	0	0	Z	0	0	0	0	0	0	0	0	0

4. Homotopy groups of some other spaces

a. Tori

n-dimensional torus can be defined as a direct product of n circles $T^n = (S^1)^{\times n}$. One can immediately derive that

$$\begin{aligned} \pi_1(T^n) &= (Z)^{\times n}, \\ \pi_k(T^n) &= 0, \quad \text{for } k \geq 2. \end{aligned}$$

b. Projective spaces

The real projective space RP^n can be represented as $RP^n = S^n/Z_2$. Therefore, $RP^1 = S^1$ and we have:

$$\begin{aligned} \pi_1(RP^1) &= Z, \\ \pi_1(RP^n) &= Z_2, \quad \text{for } n \geq 2, \\ \pi_k(RP^n) &= \pi_k(S^n), \quad \text{for } k \geq 2. \end{aligned}$$

Homotopy groups of real projective spaces												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
RP^1	Z	0	0	0	0	0	0	0	0	0	0	0
RP^2	Z_2	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^3	Z_2	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^4	Z_2	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2

Similarly for complex projective spaces CP^n we have $CP^1 = S^2$ and generally $CP^n = S^{2n+1}/S^1$. We have for homotopy groups

$$\begin{aligned} \pi_1(CP^n) &= 0, \\ \pi_2(CP^n) &= Z, \\ \pi_k(CP^n) &= \pi_k(S^{2n+1}), \quad \text{for } k \geq 3. \end{aligned}$$

Homotopy groups of complex projective spaces												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
CP^1	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
CP^2	0	Z	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}
CP^3	0	Z	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0
CP^4	0	Z	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}

[1] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry-Methods and Applications : Part II, the Geometry and Topology of Manifolds* (Graduate Texts in Mathematics, Vol 104), Springer Verlag, 1985, ISBN: 0387961623.

[2] M. Monastyrsky and O. Efimov, *Topology of gauge fields and condensed matter*, Plenum Pub., 1993.

[3] Mikio Nakahara, *Geometry, Topology, and Physics*, 3rd edition, Cambridge, Massachusetts, MIT press, 1987, ISBN: 0852740956.

[4] K. Ito, *Encyclopedic dictionary of mathematics*, 3rd edition, Cambridge, Massachusetts, MIT press (1987) Appendix A, table 6.