Exercises to lecture 1

Exercise 1

The Euclidian path integral for a particle on a ring with twisted boundary conditions is given by

$$Z = \int \mathcal{D}\phi \, e^{-\int_0^\beta d\tau \left(\frac{m\dot{\phi}^2}{2} + i\frac{\theta}{2\pi}\dot{\phi}\right)}.$$

Using the decomposition

$$\phi(\tau) = \frac{2\pi}{\beta} n\tau + \sum_{l\in\mathbf{Z}} \phi_l e^{i\frac{2\pi}{\beta}l\tau}$$

rewrite the partition function as a sum over topological sectors labeled by winding number $n \in \mathbf{Z}$ and calculate it explicitly. Find the energy spectrum from the obtained expression.

Hint: Use summation formula

$$\sum_{n=-\infty}^{+\infty} e^{-\frac{1}{2}An^2 + iBn} = \sqrt{\frac{2\pi}{A}} \sum_{l=-\infty}^{+\infty} e^{-\frac{1}{2A}(B-2\pi l)^2}.$$

Exercise 2

The classical action for a particle on a ring is given by

$$S = \int dt_p \left(\frac{m\dot{\phi}^2}{2} - \frac{\theta}{2\pi}\dot{\phi}\right),$$

where t_p is some "proper" time. Reparametrizing time as $t_p = f(t)$ we have $dt_0 = f'dt$ and $dt_0^2 = f'^2 dt^2$ and identify the metric as $g_{00} = f'^2$ and $g^{00} = f'^{-2}$. We also have $\sqrt{g_{00}} = f'$. Rewrite the action in terms of $\phi(t)$ instead of $\phi(t_p)$. Check that it has a proper form if written in terms of the introduced metric. Using the general formula for variation of the action with respect to a metric ($g = \det g_{\mu\nu}$)

$$\delta S = \int dx \sqrt{g}\, T_{\mu\nu}\delta g^{\mu\nu},$$

find the stress-energy tensor for the particle on the ring. Check that T_{00} is, indeed, the energy of the particle.

Exercise 3

Nematic is a liquid crystal characterized by an order parameter which is the unit three-component vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$ with an additional condition $\vec{n} \sim -\vec{n}$. The latter means that two unit vectors which are opposite to each other describe the same state.

What types of topological defects and textures are allowed for three-dimensional nematic? What about two-dimensional one?

Exercise 4

One can view a crystalline state as continuous translational symmetry broken to the subgroup of discrete translations. Then the order parameter space should be identified (for three-dimensional crystal) with $M = G/H = R^3/(Z \times Z \times Z)$.

a) What (geometrically) is the order parameter space for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

Exercise 5

The order parameter of superfluid $^3He - A$ can be represented by two mutually orthogonal unit vectors $\vec{\Delta}_1$, $\vec{\Delta}_2$. That is, at each point in three-dimensional space one has a pair of vectors with properties $\vec{\Delta}_1^2 = \vec{\Delta}_2^2 = 1$ and $\vec{\Delta}_1 \cdot \vec{\Delta}_2 = 0$.

a) What is the manifold of degenerate states for this system?

b) What are the homotopy groups of this manifold $\pi_{0,1,2,3}(M)$?

c) What types of topological defects and textures are allowed in such a system?

Exercise 6

What topological defects and textures one should expect in the ordered state of a three-dimensional classical Heisenberg model? What changes if the order parameter is a director instead of a vector? A "director" means a vector without an arrow, i.e., one should identify $\vec{S} \equiv -\vec{S}$. The models with a director as an order parameter are used to describe nematic liquid crystals.

Exercise 5: Topological invariant

Consider a three-dimensional unit vector field $\vec{n} \in S^2$ on a two-dimensional plane $\vec{n}(x, y)$ with constant boundary conditions $\vec{n}(x, y) \to \hat{e}_3$ as $(x, y) \to \infty$. Show that

$$Q = \int d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] \tag{1}$$

is an integer-valued topological invariant. Namely,

a) Show that under small variation $\delta\vec{n}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (1) is a Jacobian of the change of variables from x, y to a sphere $\vec{n}$ and it is normalized in such a way that the area of the sphere is 1. Therefore, Q is an integer degree of mapping of a plane (with constant boundary conditions) onto a sphere.

Hint: In b) consider the vicinity of the northern pole of the sphere only and extend your result to the whole sphere by symmetry.