# 2

# Motivating example: Particle on a ring

## 2.1 Classical particle on a ring: Action, Lagrangian, and Hamiltonian

As a simple motivating example let us consider a particle on a ring. Classically, the motion can be described by the principle of least action. A classical action $S$ of a particle can be taken as

$$S[\phi] = \int dt\, L(\phi, \dot{\phi}), \tag{2.1}$$

$$L = \frac{M}{2}\dot{\phi}^2 + A\dot{\phi} \tag{2.2}$$

should be minimal (locally) on classical trajectories. Here, the angle $\phi(t)$ is chosen to be a generalized coordinate of the particle on a ring, $M$ is a moment of inertia of a particle (or mass for a unit ring), $A$ is some constant.

Euler-Langrange equations of motion are given in terms of Lagrangian $L$ by $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$ or explicitly

$$M\ddot{\phi} = 0. \tag{2.3}$$

Particle given an initial velocity moves with constant angular velocity along the ring. Notice, that the last term of (2.2) does not affect the motion of the particle. Indeed, this term is a total time derivative and can not affect the principle of least action [1].

Given initial position of a particle on a ring at $t = t_1$ and a final position at $t = t_2$ there are infinitely many solutions of (2.3). They can be labeled by an integer number of times particle goes around the ring to reach its final position. This happens because of *nontrivial topology* of the ring – one should identify $\phi = \phi + 2\pi$ as labeling the same point on the ring. This is not very important classically as we can safely think of the angle $\phi$ taking all real values from $-\infty$ to $+\infty$. Given initial position $\phi(t_1) = \phi_1$ and initial velocity $\dot{\phi}(t_1) = \omega_1$ one can unambiguously determine the position of the particle $\phi(t)$ at all future times using (2.3).

Let us now introduce the momentum conjugated to $\phi$ as

$$p = \frac{\partial L}{\partial \dot{\phi}} = M\dot{\phi} + A \tag{2.4}$$

and the Hamiltonian as

$$H = p\dot{\phi} - L = \frac{1}{2M}(p - A)^2. \tag{2.5}$$

Corresponding Hamilton equations of motion

$$\dot{\phi} = \frac{1}{M}(p - A), \tag{2.6}$$

$$\dot{p} = 0 \tag{2.7}$$

are equivalent to (2.3).

Notice that although the parameter $A$ explicitly enters Hamiltonian formalism it only changes the definition of generalized momentum $M\dot{\phi} + A$ instead of more conventional $M\dot{\phi}$. It does not change the solution of equations of motion and can be removed by a simple canonical transformation $p \to p + A$. We will see below that this changes for a *quantum* particle.

## 2.2 Quantum particle on a ring: Hamiltonian and spectrum

Let us now consider a quantum particle on a ring. We replace classical Poisson's bracket $\{p, \phi\} = 1$ by quantum commutator $[p, \phi] = -i\hbar$ and use $\phi$-representation, i.e., we describe our states by wave functions on a ring $\psi(\phi)$. In the following we will put $\hbar = 1$. In this representation we can use $p = -i\partial_\phi$ and rewrite (2.5) as a quantum Hamiltonian

$$H = \frac{1}{2M}\left(-i\partial_\phi - A\right)^2. \tag{2.8}$$

The eigenstates and eigenvalues of this Hamiltonian are given by solutions of stationary Schrödinger equation $H\psi = E\psi$. We impose *periodic boundary conditions* requiring $\psi(\phi + 2\pi) = \psi(\phi)$, i.e., the wave function is required to be a single-valued function on the ring. The eigenfunctions and eigenvalues of (2.8) are given by

$$\psi_m = e^{im\phi}, \tag{2.9}$$

$$E_m = \frac{1}{2M}(m - A)^2, \tag{2.10}$$

where $m = 0, \pm 1, \pm 2, \ldots$ is any integer number - the quantized eigenvalue

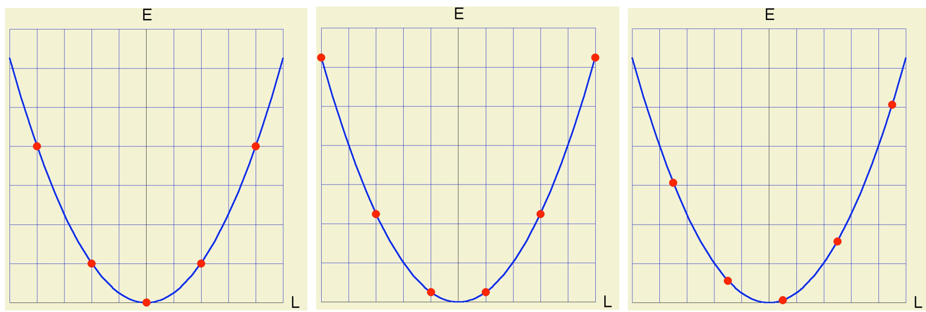


Fig. 2.1. The spectrum of the particle on a ring is shown for $2\pi A = \theta = 0, \pi, \pi/2$ respectively. The classical energy $E(p)$ is represented by a parabola and does not depend on the parameter $A$.

of the momentum operator $p = -i\partial_\phi$. We notice that although classical model is not sensitive to the parameter $A$ the quantum one is because of the quantization of $p$. The parameter $A$ can be interpreted as a vector potential of the magnetic flux penetrating the ring. This vector potential is not observable in classical mechanics but affects the quantum spectrum because of multiple-connectedness of the ring (there are many non-equivalent ways to propagate from the point 1 to the point 2 on a ring). More precisely our parameter $A$ should be identified with the vector potential multiplied by $\frac{e}{\hbar c}$. It corresponds to the magnetic flux through the ring $\Phi = A\Phi_0$, where $\Phi_0$ is a flux quantum $\Phi_0 = 2\pi\frac{\hbar c}{e}$.

The $A$-term of the classical action – *topological term* – can be written as

$$S_{top} = \int_{t_1}^{t_2} dt\, A\dot{\phi} = 2\pi A \frac{\phi_2 - \phi_1}{2\pi} = \theta \frac{\Delta\phi}{2\pi}. \tag{2.11}$$

It depends only on the initial and final values $\phi_{1,2} = \phi(t_{1,2})$ and changes by $\theta = 2\pi A$ every time particle goes a full circle around the ring in counter-clockwise direction. The conventional notation $\theta$ for a coefficient in front of this term gave the name *topological theta-term* for these type of topological terms.

The spectrum (2.10) is shown in Figure 2.1 for three values of flux through the ring $\theta = 0, \pi, \pi/2$ ($A = \Phi/\Phi_0 = 0, 1, 1/2$).

Several comments are in order. (i) an integer flux $A$-integer or $\theta$ - multiple of $2\pi$ does not affect the spectrum. (ii) There is an additional symmetry (parity) of the spectrum at $\theta$ multiple of $\pi$ (integer or half-integer flux). (iii) For half-integer flux $\theta = \pi$ the ground state is doubly degenerate $E_0 = E_1$.

Finally, let us try to remove the $A$ term by canonical transformation as in classical case. We make a gauge transformation $\psi \to e^{iA\phi}\psi$ and obtain $p \to p + A$ and $H = \frac{1}{2M}(-i\partial_\phi)^2$. One might think that we removed the effects of the $A$ term completely. However, this transformation changes the boundary conditions of the problem replacing them by *twisted boundary conditions* $\psi(\phi + 2\pi) = e^{-i2\pi A}\psi(\phi)$. The eigenfunctions satisfying twisted boundary conditions are $\psi_m = e^{i(m-A)\phi}$ and produce the same eigenvalues (2.10). We conclude that it is not possible to remove the effects of topological $A$-term in quantum mechanics. The parameter $A$ can be formally removed from the Hamiltonian by absorbing it into the boundary conditions. This, however, does not change the spectrum and other physical properties of the system.

## 2.3 Quantum particle on a ring: path integral and Wick's rotation

A quantum mechanics of a particle on a ring described by the classical action (2.2) can be represented by path integral

$$Z = \int D\phi \; e^{iS[\phi]}, \tag{2.12}$$

where integration is taken over all possible trajectories $\phi(t)$ (with proper boundary values). In this approach the contribution of the topological term to the weight in the path integral is the phase $e^{i\theta\Delta\phi/(2\pi)}$ which is picked up by a particle moving in the presence of the vector potential.

Let us perform *Wick's rotation* replacing the time by an imaginary time $\tau = it$. Then

$$\int dt \, \frac{M\dot{\phi}^2}{2} \quad \to \quad i\int d\tau \, \frac{M\dot{\phi}^2}{2}, \tag{2.13}$$

$$\int dt \, A\dot{\phi} \quad \to \quad \int d\tau \, A\dot{\phi}, \tag{2.14}$$

where in the r.h.s dot means the derivative with respect to $\tau$. The path integral (2.12) is then replaced by a Euclidian path integral

$$Z = \int_{e^{i[\phi(T)-\phi(0)]}=1} \mathcal{D}\phi \, e^{-S[\phi]}, \tag{2.15}$$

where the action

$$S = \int_0^\beta d\tau \, \left[\frac{M}{2}\dot{\phi}^2 - iA\dot{\phi}\right]. \tag{2.16}$$

We considered the amplitude of the return to the initial point in time $\beta$, i.e. $0 < \tau < \beta$. This requires periodic boundary conditions in time $e^{i\phi(0)} = e^{i\phi(\beta)}$.

We notice here that because the $A$-term is linear in time derivative it does not change its form under Wick's rotation (2.14) and therefore, is still imaginary in Euclidian formulation (2.16). Without imaginary term one can think about $e^{-S}$ as of Boltzmann weight in the classical partition function.

One can satisfy the boundary conditions as $\phi(\beta) - \phi(0) = 2\pi Q$ with any integer $Q$. We can rewrite the partition function (2.15) as:

$$Z = \sum_{Q=-\infty}^{+\infty} e^{i\theta Q} \int_{\phi(\beta)-\phi(0)=2\pi Q} \mathcal{D}\phi \, e^{-\int_0^\beta d\tau \, \frac{M}{2}\dot{\phi}^2}. \tag{2.17}$$

We notice here that $\theta = 2\pi n$ – multiple of $2\pi$ – is equivalent to $\theta = 0$. Second we notice that the partition function is split into the sum of path integrals over distinct *topological sectors* characterized by an integer number $Q$ which is called the *winding number*. The contributions of topological sectors to the total partition function are weighed with the complex weights $e^{i\theta Q}$.

For future comparisons let us write (2.16) in terms of a unit two-component vector $\vec{\Delta} = (\Delta_1, \Delta_2) = (\cos\phi, \sin\phi)$, $\vec{\Delta}^2 = 1$.

$$S = \int_0^\beta d\tau \, \left[ \frac{M}{2}\dot{\vec{\Delta}}^2 - iA(\Delta_1\dot{\Delta}_2 - \Delta_2\dot{\Delta}_1) \right]. \tag{2.18}$$

This is the simplest $(0+1)$-dimensional $O(2)$ non-linear $\sigma$-model.

## 2.4 Quantum doublet

Let us consider a particular limit of a very light particle on a circle $M \to 0$ in the presence of half of the flux quantum $A = 1/2$, $\theta = \pi$. With this flux the ground state of the system is doubly degenerate $E_0 = E_1$ and the rest of the spectrum is separated by the energies $\sim 1/M \to \infty$ from the ground state (2.10). At large $\beta$ (low temperatures) we can neglect contributions of all states except for the ground state.

We write the general form of the ground state wave function as $\alpha|+1/2\rangle + \beta|-1/2\rangle$, where $|+1/2\rangle = \psi_0$ and $|-1/2\rangle = \psi_1$. The ground state space $(\alpha, \beta)$ coincides with the one for a spin 1/2. One might say that (2.15-2.16) with $M \to 0$ realize a path integral representation for the quantum spin 1/2. This representation does not have an explicit $SU(2)$ symmetry. We will consider an $SU(2)$-symmetric path integral representation for quantum spins later.

Meanwhile, let us discuss some topological aspects of a plane rotator problem.

## 2.5 Full derivative term and topology

From a mathematical point of view the motion of a particle on the unit circle with periodic boundary conditions in time is described by a mapping $\phi(t): \quad S^1_t \to S^1_\phi$ of a circle formed by compactified time $S^1_\tau = t \in [0, \beta]$ into a circle $S^1_\phi = \phi \in [0, 2\pi]$. This mapping can be characterized by integer *winding number* $Q$ which tells us how many times the image $\phi$ goes around *target space* $S^1_\phi$ when variable $\tau$ changes from 0 to $\beta$.

It can be shown that two such mappings $\phi_1(\tau)$ and $\phi_2(\tau)$ can be continuously deformed one into another if and only if they are characterized by the same winding number. Therefore, all mappings are divided into topological classes enumerated by $Q = 0, \pm 1, \pm 2, \ldots$. Moreover, one can define a group structure on topological classes. First we define the *product* of two mappings $\phi_1$ and $\phi_2$ as

$$\phi_2 \cdot \phi_1(\tau) = \begin{cases} \phi_1(2t), & \text{for} \quad 0 < \tau < \beta/2 \\ \phi_1(\beta) + \phi_2(2\tau - \beta), & \text{for} \quad \beta/2 < \tau < \beta. \end{cases}$$

If $\phi_1$ belongs to the topological class $Q_1$ and $\phi_2$ to $Q_2$, their product belongs to the class $Q_1 + Q_2$. One can say that the product operation on mappings induces the structure of Abelian group on the set of topological classes. In this case this group is the group of integer numbers with respect to addition. One can write this fact down symbolically as $\pi_1(S^1) = Z$, where subscript one denotes that our time is $S^1$ and $S^1$ in the argument is our target space. One says that the *first (or fundamental) homotopy group of $S^1$ is the group of integers.*

There is a simple formula giving the topological class $Q \in Z$ in terms of $\phi(\tau)$

$$Q = \int_0^\beta \frac{d\tau}{2\pi} \, \dot{\phi}. \tag{2.19}$$

Let us now assume that we splitted our partition function into the sum over different topological classes. What are the general restrictions on the possible complex weights which one can introduce in the physical problem. One can deform smoothly any mapping in the class $Q_1 + Q_2$ into two mappings of classes $Q_1$ and $Q_2$ which are separated by a long time. Because of the multiplicative property of amplitudes this means that the weights $W_Q$ associated with topological classes must form a (unitary) representation of the fundamental group of a target space. The only unitary representation of $Z$ is given by $W_Q = e^{i\theta Q}$ with $0 < \theta < 2\pi$ labelling different representations. In the case of plane rotator these weights correspond to the magnetic flux piercing the one-dimensional ring.

In more general case of, say, particle moving on the *manifold* $G$ (instead of $S^1$) we have to consider the fundamental group of our target space $\pi_1(G)$, find its unitary representations, and obtain complex weights which could be associated with different topological classes.

### 2.6 Topological terms and quantum interference

As it can be seen from (2.17) the presence of a topological term in the action ($\theta \neq 0$) results in the interference between topological sectors in the partition function. The Boltzmann weight calculated for a trajectory within a given topological sector $Q$ is additionally weighted with complex phase $e^{i\theta Q}$. This interference can not be removed by Wick's rotation.

### 2.7 General definition of topological terms

We define generally *topological terms* as the *metric-independent* terms in the action.

A universal object present in any field theory, is the symmetric *stress-energy tensor* $T_{\mu\nu}$. It can be defined as a variation of the action with respect to the metric $g^{\mu\nu}$. More precisely, an infinitesimal variation of the action can be written as

$$\delta S = \int dx \sqrt{g}\, T_{\mu\nu} \delta g^{\mu\nu}, \tag{2.20}$$

where $\sqrt{g}\, dx$ is an invariant volume of space-time.

It immediately follows from our definition of topological terms that topological terms do not contribute to the stress-energy tensor. If in a field theory all terms are topological we have $T_{\mu\nu} = 0$ for such a theory. These theories are called topological field theories.

A particular general covariant transformation is the rescaling of time. Topological terms do not depend on a time scale. Therefore, the corresponding Lagrangians are linear in time derivatives. They do not transform under Wick's rotation and are always imaginary in Euclidian formulation. They describe quantum interference which is not removable by Wick rotation.

### 2.8 Theta terms and their effects on the quantum problem

Theta terms are topological terms of particular type. They appear when there exist nontrivial topological textures in space-time. Essentially, these terms are just complex weights of different topological sectors in the path integration. We will go over more details on $\theta$-terms later in the course.

In addition to being imaginary in Euclidian formulation as all other topological terms $\theta$-terms have also some special properties. These properties distinguish them from other types of topological terms. The following is a partial list of the features of topological $\theta$-terms and of their manifestations.

- Textures in space-time (integer topological charge $Q$)
- Realize irreducible 1d-representations of $\pi_D(G)$, where $D$ is the dimension of space-time and $G$ is a target space
- Quantum interference between topological sectors
- Do not affect equations of motion
- Affect the spectrum of quantum problem by changing quantization rules
- Periodicity in coupling constant $\theta$
- $\theta$ is not quantized (for $Q \in Z$)
- $\theta = 0, \pi$ an additional (parity) symmetry
- $\theta = \pi$ – degeneracy of the spectrum. Gapless excitations.
- Equivalent to changes in boundary conditions.
- $\theta$ is a new parameter which appears from the ambiguity of quantization of the classical problem for multiply-connected configurational space.

## 2.9 Exercises

**Exercise 2.1: Particle on a ring, path integral** The Euclidian path integral for a particle on a ring with magnetic flux through the ring is given by

$$Z = \int \mathcal{D}\phi \; e^{-\int_0^\beta d\tau \left(\frac{m\dot{\phi}^2}{2} - i\frac{\theta}{2\pi}\dot{\phi}\right)}.$$

Using the decomposition

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{l\in\mathbf{Z}} \phi_l e^{i\frac{2\pi}{\beta}l\tau}$$

rewrite the partition function as a sum over topological sectors labeled by winding number $Q \in \mathbf{Z}$ and calculate it explicitly. Find the energy spectrum from the obtained expression.

*Hint*: Use summation formula

$$\sum_{n=-\infty}^{+\infty} e^{-\frac{1}{2}An^2 + iBn} = \sqrt{\frac{2\pi}{A}} \sum_{l=-\infty}^{+\infty} e^{-\frac{1}{2A}(B-2\pi l)^2}.$$

**Exercise 2.2: spin 1/2 from a particle on a ring** Calculate the partition function of a particle on a ring described in the previous exercise. Find explicit expressions in the limit $M \to 0$, $\theta \to \pi$ but $\theta - \pi \sim M/\beta$. One can interpret the obtained partition function as a partition function of a spin 1/2. What is the physical meaning of the ratio $(\theta-\pi)/M$ in the spin 1/2 interpretation of the result?

**Exercise 2.3: Metric independence of the topological term** The classical action for a particle on a ring is given by

$$S = \int dt_p \left( \frac{m\dot{\phi}^2}{2} - \frac{\theta}{2\pi}\dot{\phi} \right),$$

where $t_p$ is some "proper" time. Reparametrizing time as $t_p = f(t)$ we have $dt_0 = f'dt$ and $dt_0^2 = {f'}^2 dt^2$ and identify the metric as $g_{00} = {f'}^2$ and $g^{00} = {f'}^{-2}$. We also have $\sqrt{g_{00}} = f'$. Rewrite the action in terms of $\phi(t)$ instead of $\phi(t_p)$. Check that it has a proper form if written in terms of the introduced metric. Using the general formula for variation of the action with respect to a metric ($g = \det g_{\mu\nu}$)

$$\delta S = \int dx \sqrt{g}\, T_{\mu\nu} \delta g^{\mu\nu},$$

find the stress-energy tensor for the particle on the ring. Check that $T_{00}$ is, indeed, the energy of the particle.