(1) $n = 3\times10^{18}\,\text{cm}^{-3}$ $\qquad k_F = (3\pi^2 n)^{1/3} = 4.5\times10^{6}\,\text{cm}^{-1}$

$m = 0.067\,m_e$

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\left[(1.054\times10^{-34}\,\text{Js})(4.5\times10^{8}\,\text{m}^{-1})\right]^2}{2(.067)(0.911\times10^{-30}\,\text{kg})}$$

$$= 1.8\times10^{2}\times10^{-52}\times10^{30} = \frac{1.8\times10^{-20}\,\text{J}}{1.6\times10^{-19}\,\text{J/eV}} =$$

$$\boxed{\begin{aligned} k_F &= 4.5\times10^{8}\,\text{m}^{-1} = .045\,\text{Å}^{-1} \\ E_F &= 0.11\,\text{eV} \\ T_F &= 1300\,\text{K} \end{aligned}}$$

(2) $\psi_k(r)$ has the fundamental, defining property $\psi_k(r+l) = e^{ik\cdot l}\psi_k(r)$

We can always define u_k by

$$\psi_k = e^{ik\cdot r}u_k(r)$$

Then
$$\psi_k(r+l) = e^{ik\cdot l}e^{ik\cdot r}u_k(r+l)$$
$$= e^{ik\cdot l}\psi_k(r)$$
$$= e^{ik\cdot l}e^{ik\cdot r}u_k(r)$$

Therefore $u_k(r+l) = u_k(r)$

$$\left(p^2/2m + V(r) - E_k\right)\psi_k = 0 = \left(p^2/2m + V(r) - E_k\right)e^{ik\cdot r}u_k$$

now
$$p\left(e^{ik\cdot r}u_k\right) = \hbar k e^{ik\cdot r}u_k + e^{ik\cdot r}pu_k$$
$$= e^{ik\cdot r}(p+\hbar k)u_k$$

$$\therefore \quad e^{ik\cdot r}\left[(p+\hbar k)^2/2m + V(r) - E_k\right]u_k = 0 \qquad \text{QED}$$

so
$$\mathcal{H}_{eff} = \boxed{\mathcal{H}_k = (p+\hbar k)^2/2m + V(r)}$$

② continued

$$\left[(p+\hbar k)^2/2m + V(r)\right] u_k(r) = \epsilon_k u_k$$

$$\mathcal{H}_0 u_n = \left[p^2/2m + V(r)\right] u_n(r) = \epsilon_n(r)$$

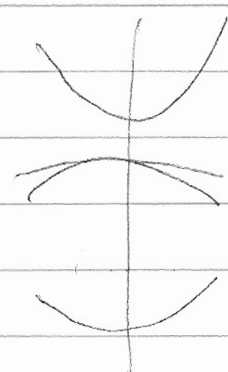
$$\Big[\mathcal{H}_0 + \underbrace{\hbar k\cdot p/m \quad + k^2/2m}_{\mathcal{H}^1}\Big] u_{nk} = \epsilon_{nk} u_{nk}$$

$$\epsilon_{nk} = \epsilon_{n0} + \langle n|\mathcal{H}'|n\rangle + \sum_m^{\neq n} \frac{|\langle n|\mathcal{H}'|m\rangle|^2}{\epsilon_n - \epsilon_m}$$

Now

$$= \epsilon_{n0} + \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2}{m^2} k_\alpha k_\beta \sum_m \frac{\langle n|p_\alpha|m\rangle\langle m|p_\beta|n\rangle}{\epsilon_n - \epsilon_m}$$

$$\frac{1}{m^*_{\alpha\beta}} = \frac{\delta_{\alpha\beta}}{m} + \frac{1}{m^2}\sum_m \frac{\langle n|p_\alpha|m\rangle\langle m|p_\beta|n\rangle}{\epsilon_n - \epsilon_m}$$



③

$$\langle kA|H|kA\rangle = \epsilon_A$$

$$\langle kB|H|kB\rangle = \epsilon_B$$

$$\langle kA|H|kB\rangle = -t\left[e^{ik_x a/2} \cdots + e^{-ik_z a/2}\right]$$

Eigenvalues are

$$\left(\frac{\epsilon_A+\epsilon_B}{2}\right) \pm \sqrt{\left(\frac{\epsilon_A-\epsilon_B}{2}\right)^2 + \left(2t\cos\frac{k_x a}{2} + 2t\cos\frac{k_y a}{2} + 2t\cos\frac{k_z a}{2}\right)^2}$$

$$\frac{\epsilon(k)}{t} = \pm 2\sqrt{1 + \left(\cos\frac{k_x a}{2} + 1 + 1\right)^2}$$