

Due date was Wednesday October 24. Here are the answers.

1. Curie Law. The figure is from S. Araj's and R. V. Colvin, J. Appl. Phys. **33**, 2517 (1962). Gd^{3+} has a $4f^7$ configuration, with approximately 7 Bohr magnetons of moment per ion. The data agree fairly well with the Curie Law. Derive the Curie Law.

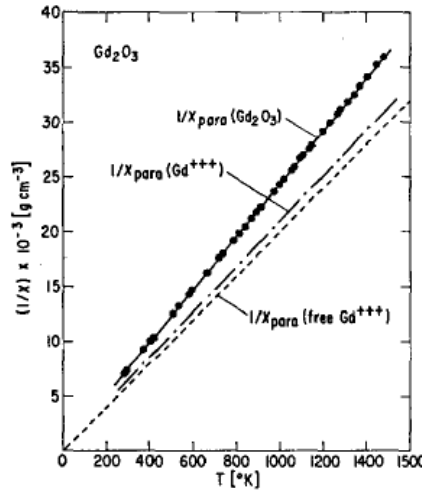


FIG. 1. Inverse magnetic susceptibility of Gd_2O_3 as a function of temperature.

Answer:

Start with $H = -\sum_i \vec{\mu}_i \cdot \vec{B}$.

The susceptibility is defined as $\vec{M} = N\langle \vec{\mu} \rangle / V = \chi \vec{B}$. The spins

are independent so we can find the answer for a single spin, and multiply by N. The partition function for a classical spin is $Z = \int d\Omega \exp(-\mu B \cos \theta / k_B T)$, where the angle θ is measured from B which is chosen as the z direction. The mean value of μ is then $\langle \mu_z \rangle = k_B T d \ln Z / dB$. To find χ , we need to know $\langle \mu_z \rangle$ only to second order in B.

Therefore we use the Taylor series of the exponential, and find

$$Z = \int d\Omega \left[1 - (\mu B \cos \theta / k_B T) + (\mu B \cos \theta / k_B T)^2 / 2 + \dots \right] = 4\pi \left[1 + (\mu B / k_B T)^2 / 6 + \dots \right].$$

From this we get immediately $\chi = N\mu^2 / 3Vk_B T$. A slightly simpler procedure is to do the quantum treatment, where the integral for Z becomes a sum over the discrete z-components of the spin. In the limit where there are many components, the classical answer is retrieved. In the limit where $s=1/2$, there are only two components, and the answer is larger than this by 3.

2. The Curie law should work only when spins on neighboring ions don't interact. Quantum mechanics allows a strong exchange interaction which is apparently not important in Gd_2O_3 . But there must always be a dipole-dipole classical interaction. As an estimate of the temperature below which the Curie law should fail, calculate the optimum attractive energy of (dipole-dipole) interaction of two Gd moments on nearest neighbor sites in Gd_2O_3 . Estimate the spacing from the density $\rho = 8.33 \text{ g/cm}^3$.

Answer:

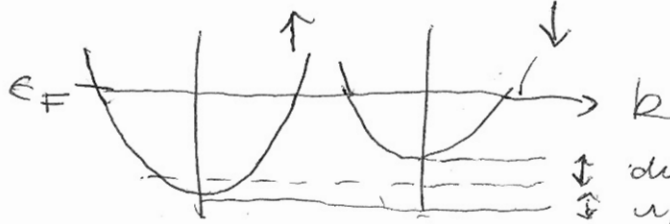
The dipole-dipole interaction for two spins separated by r is

$$V = \frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{r^3} - \frac{3(\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r})}{r^5}$$

and has optimum value $V = -(\mu_0 / 4\pi)2\mu^2 / r^3$ when the spins are parallel and point along the axis separating the spins. The factor $(\mu_0 / 4\pi)$ is needed to convert to SI

units. The factor $1/r^3$ can be approximated by n , the number of Gd atoms per unit volume, which is the mass density, $8.33 \times 10^3 \text{ kg/m}^3$, divided by the number of kg per $\text{GdO}_{1.5}$ fragment. Each $\text{GdO}_{1.5}$ has mass 181.25 amu, and there are $1.66 \times 10^{-27} \text{ kg}$ per amu. This gives a separation $r = 3.31 \text{ \AA}$. If we take μ to be 7 Bohr magnetons, and convert using $1 \text{ Bohr magneton} = 0.927 \times 10^{-23} \text{ J/T}$, and $(\mu_0 / 4\pi) = 10^{-7} \text{ T}^2\text{m}^3/\text{J}$, the answer is $|V/k_B| = 1.7 \text{ K}$. At temperatures of order 1 K, the dipole-dipole interaction begins to have serious consequences.

3. In metals with weak exchange interactions between spins, the susceptibility (as explained by Pauli) is smaller than the Curie law by the small factor $\sim k_B T / E_F$. Derive the Pauli susceptibility. Assume an ordinary metal with a density of states $D(E)$ which varies slowly with E on a scale of $k_B T$.



↓ down spins energy raised by μ_B
 ↑ up spins energy lowered by μ_B

$$N = \# \text{electrons} = \int_{-\infty}^{E_F} dE \left[D_{\uparrow}(E) f(E) + D_{\downarrow}(E) f(E) \right]$$

$$= \# \text{electrons in a magnetic field} = \int_{-\infty}^{E_F + \mu_B} dE D_0(E) + \int_{-\infty}^{E_F - \mu_B} dE D_0(E)$$

$$= 2 \int_{-\infty}^{E_F} dE D_0(E)$$

Since $D_0(E) = \text{density of states} = \sum_{\vec{k}} \delta(E_{\vec{k}} - E)$
 for one spin orientation $= D_{\uparrow}(E) = D_{\downarrow}(E) \approx D(E_F)$
 is approximately constant on an energy scale of $\mu_B B$,
 these equations are satisfied automatically. E_F does not shift with B .

$$M = \frac{\mu_B}{V} \left(\int_{-\infty}^{E_F + \mu_B} dE - \int_{-\infty}^{E_F - \mu_B} dE \right) D_0(E) \approx \frac{2}{V} D_0(E_F) \mu_B^2 B$$

$$\chi = \frac{2}{V} D_0(E_F) \mu_B^2 = \left[\frac{1}{V} D(E_F) \mu_B^2 = \chi \right]$$

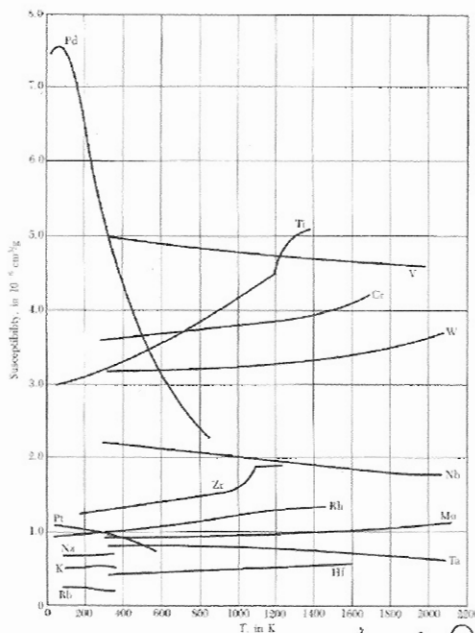
where $D(E_F) = 2 D_0(E_F)$ is the density of states for both spin orientations,

Note: $\frac{\chi_{\text{Pauli}}}{\chi_{\text{Curie}}} \approx \frac{D_0(E_F) \cdot k_B T}{N} \sim \frac{k_B T}{E_F}$

since $D_0(E_F) \sim \frac{N}{E_F}$ ($\frac{3}{2} \frac{N}{E_F}$ for free electrons)

4. Aluminum metal has a Pauli susceptibility which is quite close to the free electron value. Suppose you put a gram of Al near a strong magnet, in a region with $B = 1 \text{ T}$ and $dB/dx = 100 \text{ T/m}$. How large is the force (compared with mg , for example.)? You may use the free electron approximation.

Note: The figure to the right is from C. Kittel, *Introduction to Solid State Physics* (6th edition, Ch 14 – J. Wiley & Sons, 1986). It shows the susceptibility of various metals to rather high T , with some mild deviations from the T -independent Pauli form, except for Pd which deviates fairly strongly. It is not accidental that Pd also has the largest low T value.



The energy in a B -field is $-\int B dM = -\frac{1}{2} \chi B^2$

and $F = \frac{d}{dx} \frac{1}{2} \chi B^2 = \chi B \frac{dB}{dx}$

$$\chi = \mu_B^2 \mathcal{D}(\epsilon_F) \approx \mu_B^2 \cdot \frac{3N_{el}}{2\epsilon_F}$$

$$Mg = \text{weight} = n \text{ Mat } g$$

$$N_{el} = 3N_{at} = 3N_A n$$

$n = \# \text{ moles Mat} = 27 \frac{\text{g}}{\text{mole}}$

$N_A = \text{Avogadro number}$

$$\frac{F}{mg} = \frac{\mu_B^2 \cdot \frac{9 N_A n}{2 \epsilon_F} B \frac{dB}{dx}}{\frac{1}{2} \text{Mat } g} \quad \epsilon_F = 11.7 \text{ eV (Al)}$$

$$= \frac{(0.927 \frac{\text{J}}{\text{T}})^2 \times 9 \times 6.023 \times 10^{23} [B = 1 \text{ T}] [\frac{dB}{dx} = 10^2 \frac{\text{T}}{\text{m}}]}{2 (11.7 \text{ eV}) (1.602 \times 10^{-19} \frac{\text{J}}{\text{eV}}) (27 \frac{\text{g}}{\text{mole}}) (9.8 \text{ m/s}^2) 10^{-3} \frac{\text{kg}}{\text{g}}}$$

$$\frac{F}{mg} = 0.047$$