1. Kittel, Ch6 problem 1, plus extra. (a) Show that the kinetic energy of a three-dimensional gas of N free electrons at 0K is $3N\varepsilon_F/5$. (b) Compute the value for an ordered Cu-Al 50/50 alloy with valence assignments 1 for Cu and 3 for Al. The alloy crystal structure is CsCl (simple cubic lattice, two atoms per cell) with lattice constant $a=2.91\ \text{Å}$. (c) Each electron has spin $\frac{1}{2}$ and magnetic moment $\mu_B = 1\ \text{Bohr magneton}$. Find the formula for the kinetic energy of an electron gas with $(1+\varepsilon) N/2$ up spins and $(1-\varepsilon) N/2$ down spins, in terms of the Fermi energy of the unpolarized gas, where $|\varepsilon| < \frac{1}{2}$. (d) Taylor-expand for small $\varepsilon$. Remember that the energy can be written as $U=U_0 + n\chi^{-1}M^2/2$. What do you get for the susceptibility $\chi$?

2. Kittel, Ch6, problem 5. Liquid $^3\text{He}$. The atom $^3\text{He}$ has spin $\frac{1}{2}$ and is a Fermion. The density of liquid He is 0.081 g/cm$^3$ near absolute zero. (a) Calculate the Fermi wave-vector, the Fermi energy and the Fermi temperature. (b) Calculate the pressure that would be needed to maintain this density (at $T=0$) if $^3\text{He}$ were a non-interacting Fermi gas. (c) What holds $^3\text{He}$ in the liquid state at $p=T=0$?

3. The simplest excited state of the non-interacting electron system is an “electron-hole pair” excitation. The ground state is a filled “Fermi sea” shown shaded. This “sea” contains $N$ electrons ($N/2$ with spin up, $N/2$ with spin down.) The excited states might involve moving an electron out of the sea and flipping its spin in the process. That would be a “spin-flip” excitation. But the most common excitation has no spin flip. For a fixed $Q$ (shown in the picture) there are a finite number of such excitations. Here we consider only the spin-conserving excitations where the electron is moved from $k$ to $k+Q$ without altering its spin. (a) Show that, for $|Q| > 2k_F$, the number of such excitations is $N$. (b) For $|Q| < 2k_F$, the number of such excitations is $3\left(\frac{|Q|}{2k_F}\right)^2\left(1 - \frac{|Q|^2}{12k_F^2}\right)N$. Use a simple argument for small $|Q|$ to argue that the approximate answer is $3\left(\frac{|Q|}{2k_F}\right)^2N$.

4. Charge density. An appropriately anti-symmetrized eigenstate of the non-interacting electron gas can be written in various ways, for example:

$$\Psi(r_1,\ldots,r_N) = \frac{1}{\sqrt{N!}} \sum_p (-1)^p \prod_{i=1}^N \psi_{p(i)}(r_i) = |l_1, l_2, \ldots, l_N, 0, 0_{N+1}, \ldots\rangle$$

where the occupied single-particle ortho-normal states $\psi_i$ are labeled from $i = 1$ to $N$. This state is normalized by $\int dr_1 \cdots \int dr_N |\Psi(r_1,\ldots,r_N)|^2 = 1$. If the single-particle energies $\varepsilon_i$ of these states are the lowest possible, then this is the ground state. $P(i)$ means a
permutation of the indices $i$, and $P$ in the exponent is the “order” (+ or − 1 for even or odd) of the permutation. This state is referred to as a “single” “Slater determinant” state. The notation $r_i$ is short for coordinate $\vec{r}_i$ and spin $\sigma_i$. The integral $dr_i$ is a 3-d spatial integral $d\vec{r}_i$, and a spin trace (1 if spins are parallel, 0 otherwise.) The electron charge density is defined as $\rho(r) = C \int dr_1 \cdots \int dr_N |\Psi(r_1, \cdots, r_N)|^2$. Show that the charge density of a single Slater determinant eigen-state is $\rho(r) = C' \sum_{i=1}^{N} |\psi_i(r)|^2$. (This is not necessarily the ground state.) Why should $C'$ be chosen to be 1? What is therefore the right choice for $C$?