In BCS theory we need eigenvalues and eigenvectors of a matrix of the type

$$
\hat{M} = \begin{pmatrix}
\xi & \Delta \\
\Delta^* & -\xi
\end{pmatrix}
$$

where $\xi = \epsilon_k - \mu$ and $\Delta$ is the complex gap, $|\Delta| \exp(i\phi)$. Clearly the eigenvalues are $\pm E$ where $E = \sqrt{\xi^2 + \Delta^2}$. It is convenient to express this matrix in terms of the Pauli matrices

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

The matrix becomes

$$
\hat{M} = E(\vec{r} \cdot \vec{\sigma})
$$

where the unit vector $\vec{r}$ is given by

$$
\vec{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
$$

and $\theta$ is defined in the picture below.

The matrix $\vec{r} \cdot \vec{\sigma}$ can be rotated until $\vec{r}$ is $z$. The eigenvalues are thus $\pm 1$. The rotation matrix $U$ is defined as

$$
\sigma_z = U(\vec{r} \cdot \vec{\sigma}) U^\dagger
$$

and is the product of two simple rotations, $U_2 U_1$, where

$$
U_1 = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = e^{i(\phi/2)\sigma_z}, \quad U_2 = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = e^{i(\theta/2)\sigma_y}.
$$

The rotation $U_1$ is around the $z$-axis by angle $-\phi$. This causes the vector $\vec{r}$ to lie in the $xz$ plane. The rotation $U_2$ is around the $y$ axis by angle $-\theta$. This causes the vector $\vec{r}$ to line up with the $z$ axis. The resulting conjugate rotation matrix $U^\dagger$

$$
U^\dagger = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} & -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}
$$

contains as its columns the two orthonormal eigenvectors $|1\rangle$ and $|-1\rangle$ of $\vec{r} \cdot \vec{\sigma}$. The eigenvectors could, of course, each be multiplied by an additional overall phase factor $\exp(i\psi_1)$ and $\exp(i\psi_{-1})$.