Physics 555 Fall 2004 (instructor: P. B. Allen)
2×2 matrix algebra for BCS theory

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In BCS theory we need eigenvalues and eigenvectors of a matrix of the type

$$\hat{M} = \begin{pmatrix} \xi & \Delta^* \\ \Delta & -\xi \end{pmatrix}$$

where $\xi = \epsilon_k - \mu$ and Δ is the complex gap, $|\Delta|\exp(i\phi)$. Clearly the eigenvalues are $\pm E$ where $E = \sqrt{\xi^2 + |\Delta|^2}$. It is convenient to express this matrix in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

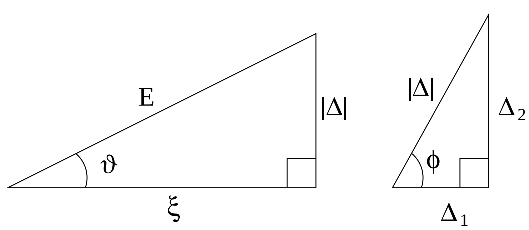
The matrix becomes

$$\hat{M} = E(\hat{r} \cdot \vec{\sigma})$$

where the unit vector $\hat{r}$ is given by

$$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

and θ and ϕ are defined in the picture below.



The matrix $\hat{r} \cdot \vec{\sigma}$ can be rotated until $\hat{r}$ is $\hat{z}$. The eigenvalues of $\hat{r} \cdot \vec{\sigma}$ are thus ± 1. The rotation matrix U is defined as

$$\sigma_z = U(\hat{r} \cdot \vec{\sigma})U^\dagger \quad \text{or} \quad \hat{r} \cdot \vec{\sigma} = U^\dagger \sigma_z U$$

and is the product of two simple rotations, $U_2 U_1$, where

$$U_1 = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = e^{i(\phi/2)\sigma_z} \quad U_2 = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = e^{i(\theta/2)\sigma_y}.$$

The rotation U_1 is around the z-axis by angle $-\phi$. This causes $\hat{r} \cdot \vec{\sigma}$ to rotate such that the new vector $\hat{r}'$ lies in the xz plane. The rotation U_2 is around the y axis by angle $-\theta$. This causes the new vector $\hat{r}''$ to line up with the z axis. The resulting conjugate rotation matrix $U^\dagger$

$$U^\dagger = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} & -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} & \cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$$

contains as its columns the two orthonormal eigenvectors $|1>$ and $|-1>$ of $\hat{r} \cdot \vec{\sigma}$. The eigenvectors could, of course, each be multiplied by an additional overall phase factor $\exp(i\psi_1)$ and $\exp(i\psi_{-1})$.