In BCS theory we need eigenvalues and eigenvectors of a matrix of the type

\[ \hat{M} = \begin{pmatrix} \xi & \Delta^* \\ \Delta & -\xi \end{pmatrix} \]

where \( \xi = \epsilon_k - \mu \) and \( \Delta \) is the complex gap, \( |\Delta| \exp(i\phi) \). Clearly the eigenvalues are \( \pm E \) where \( E = \sqrt{\xi^2 + |\Delta|^2} \). It is convenient to express this matrix in terms of the Pauli matrices

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

The matrix becomes

\[ \hat{M} = E (\hat{r} \cdot \hat{\sigma}) \]

where the unit vector \( \hat{r} \) is given by

\[ \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) , \]

and \( \theta \) and \( \phi \) are defined in the picture below.

The matrix \( \hat{r} \cdot \hat{\sigma} \) can be rotated until \( \hat{r} \) is \( \hat{z} \). The eigenvalues of \( \hat{r} \cdot \hat{\sigma} \) are thus \( \pm 1 \). The rotation matrix \( U \) is defined as

\[ \sigma_z = U (\hat{r} \cdot \hat{\sigma}) U^\dagger \quad \text{or} \quad \hat{r} \cdot \hat{\sigma} = U^\dagger \sigma_z U \]

and is the product of two simple rotations, \( U_2 U_1 \), where

\[
U_1 = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \quad U_2 = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} = e^{i(\phi/2)|\Delta|}.
\]

The rotation \( U_1 \) is around the \( z \)-axis by angle \( -\phi \). This causes \( \hat{r} \cdot \hat{\sigma} \) to rotate such that the new vector \( \hat{r}' \) lies in the \( xx \) plane. The rotation \( U_2 \) is around the \( y \) axis by angle \( -\theta \). This causes the new vector \( \hat{r}'' \) to line up with the \( z \) axis. The resulting conjugate rotation matrix \( U^\dagger \)

\[
U^\dagger = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} & -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}
\]

contains as its columns the two orthonormal eigenvectors \( |1> \) and \( |-1> \) of \( \hat{r} \cdot \hat{\sigma} \). The eigenvectors could, of course, each be multiplied by an additional overall phase factor \( \exp(i\psi_1) \) and \( \exp(i\psi_{-1}) \).