9/13/01. **Due:** Tuesday, September 25, 2001 (in class).

Physics 503: Methods of Mathematical Physics

Homework 2

Exercise 1

Show that the cross-ratio is an invariant of fractional transformation, i.e., that

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

when $w = \frac{az+b}{cz+d}$.

Exercise 2 (CKP, page 19, problem 1)

Use the cross-ratio to obtain a mapping which transforms the upper-half z plane into the interior of the unit circle in the w plane. Sketch the w images of vaious points and curves in the z plane, and vice versa. (*Hint*: Replace z_1, w_1 by z, w; set z_2, z_3, z_4 equal to $-1, 0, 1$, etc. Or use point at infinity.)

Exercise 3

Discuss the branch cut and Riemann-surface situation for the following function

$$g(z) = \sqrt{1 + \sqrt{z}}.$$

Exercise 4

Verify the Cauchy-Riemann equations for $(1 - z^2)^{1/2}$. At what points this function has singularities?

Exercise 5

Prove in an easy way that $(x^2 + y^2)^{1/4} \cos\left(\frac{1}{2} \arctan \frac{y}{x}\right)$ is harmonic.

Exercise 6 (CKP, page 36, problem 3)

Show in an easy way that the integral of each of the following expressions around the circle $|z| = 1/2$ vanishes:

$$a) \quad \frac{z+1}{z^2+z+1}, \qquad b) \quad e^{z^2} \ln(1 + z), \qquad c) \quad \arcsin z.$$

Exercise 7 (CKP, page 40, problem 1)

Use Cauchy's integral formula to evaluate the integral around the unit circle ($|z| = 1$) of

$$a) \quad \frac{\sin z}{2z+i}, \qquad b) \quad \frac{\ln(z+2)}{z+2}, \qquad c) \quad \frac{z^3+\operatorname{arcsinh}(z/2)}{z^2+iz+i}.$$