Physics 503: Methods of Mathematical Physics

Homework 7

Exercise 1 (FS 81.1cg)

Evaluate the following integrals

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^4}\, dx,$$
$$I = \int_{-1}^{1} \frac{dx}{[(1-x)(1+x)^2]^{1/3}}.$$

Exercise 2

Find the leading behavior (both exponent and pre-exponential factor) of the integral $I(\omega) = \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(\cosh t)^{2/3}}\, dt$ as $\omega \to +\infty$.

Exercise 3 (BO 6.74)

Find three terms in the asymptotic behavior of $I(x) = \int_0^1 \ln(1+t) e^{ix\sin^2 t}\, dt$ as $x \to +\infty$.

Exercise 4 (BO 6.56abc)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$a)\quad I(x) = \int_0^1 e^{ixt^2} \cosh t^2\, dt,$$
$$b)\quad I(x) = \int_0^1 \cos(xt^4) \tan t\, dt,$$
$$c)\quad I(x) = \int_0^1 e^{ix(t-\sin t)}\, dt.$$

Exercise 5 (BO 6.92b)

Find the leading behavior of the sum $S(x) = \sum_{k=0}^{\infty} \frac{1}{(k^2+x^2)^2}$ as $x \to +\infty$.

Exercise 6 (BO 6.93ac)

Find three terms in the asymptotic behavior as $n \to +\infty$ of the following sums:

$$
\begin{aligned}
a) \quad S_n &= \sum_{k=1}^{n} \frac{(-1)^k}{k}, \\
b) \quad S_n &= \sum_{k=1}^{n} \frac{\sin k}{k}.
\end{aligned}
$$

Exercise 7 (FS 81.12)

Express the following integrals in terms of the Γ-function:

$$
\begin{aligned}
a) \quad I &= \int_0^{\pi/2} \sin^{2p}\phi \cos^{2q}\phi \, d\phi, \\
b) \quad I &= \int_0^{\pi/2} \tan^{p}\phi \, d\phi, \\
c) \quad I &= \int_0^{1} \frac{dx}{\sqrt{1-x^4}}, \\
d) \quad I &= \int_0^{+\infty} \frac{x^m \, dx}{(a+bx^n)^p},
\end{aligned}
$$

where $a > 0$, $b > 0$, $np > m + 1$.

The problems referred to as BO are taken from the book:

Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, Inc., New York, 1978.

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.