Physics 503: Methods of Mathematical Physics

Homework 8

Exercise 1

Evaluate the integral $\int_0^1 \psi(x)\sin(2\pi n x)\,dx$ for $n = 0, 1, 2, \ldots$. Here $\psi(x)$ is the digamma function.

Exercise 2

Evaluate the integral (exactly)

$$\int \frac{d^2p}{(2\pi)^2}\,\frac{e^{i\vec{p}\cdot\vec{r}}}{p^2+m^2},$$

where the integral is taken over two-dimensional p-plane, $\vec{p}\cdot\vec{r} = p_1 x + p_2 y$, $p = \sqrt{p_1^2 + p_2^2}$, and m is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as $r \to \infty$. Find the leading asymptotic behavior of the result as $r \to 0$. Can you obtain the latter not from the exact result but directly from the integral?

Exercise 3

Find three terms of an asymptotic expansion of Macdonald's function $K_0(x)$ as $x \to \infty$ starting from the integral representation

$$K_0(x) = \int_0^\infty e^{-z\cosh t}\,dt.$$

Exercise 4

Calculate the following integral (exactly)

$$\int_0^\infty e^{-ax} J_n(bx)\,dx.$$

Hint: You can use the generating function for Bessel functions. First calculate the integral over x and then find the coefficient in Laureant series in t using the closed contour integral in t-plane.

Exercise 5 (CKP, page 230, problem 7a)

Show that

$$K_\nu(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_\nu(z) = \frac{1}{z}.$$

Hint: Use recurrence relations for I and K derived from the ones for J.

Exercise 6 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$\begin{aligned} a) \quad I(x) &= \int_0^1 \sin\left[x\left(t + \frac{t^3}{6} - \sinh t\right)\right] dt, \\ b) \quad I(x) &= \int_{-1}^1 \sin\left[x(t - \sin t)\right] \sinh t \, dt. \end{aligned}$$