

Physics 503: Methods of Mathematical Physics

Read: CKP sections 5-1, 5-5, 6-4.

BO sections 6.5.

“**CKP**” refers to Carrier, Krook, and Pearson book.

“**BO**” refers to Bender and Orszag book.

Problems with stars are not for credit and will NOT be graded.

Homework 7

*Exercise 1

Evaluate the integral $\int_0^1 \psi(x) \sin(2\pi nx) dx$ for $n = 0, 1, 2, \dots$. Here $\psi(x)$ is the digamma function.

Exercise 2

Evaluate the integral (exactly)

$$\int \frac{d^2p}{(2\pi)^2} \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2 + m^2},$$

where the integral is taken over two-dimensional p -plane, $\vec{p}\cdot\vec{r} = p_1x + p_2y$, $p = \sqrt{p_1^2 + p_2^2}$, and m is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as $r \rightarrow \infty$. Find the leading asymptotic behavior of the result as $r \rightarrow 0$. Can you obtain the latter not from the exact result but directly from the integral?

Exercise 3

Find three terms of an asymptotic expansion of Macdonald's function $K_0(x)$ as $x \rightarrow \infty$ starting from the integral representation

$$K_0(x) = \int_0^\infty e^{-x \cosh t} dt.$$

*Exercise 4

Calculate the following integral (exactly)

$$\int_0^\infty e^{-ax} J_n(bx) dx.$$

Hint: You can use the generating function for Bessel functions. First calculate the integral over x and then find the coefficient in Laurent series in t using the closed contour integral in t -plane.

Exercise 5

Calculate the following integral (exactly)

$$\int_0^{\infty} \frac{e^{-px} dx}{\sqrt{x(x+a)}}.$$

Here $a > 0, p > 0$.

Hint: The answer is given in terms of Macdonald's function.

Exercise 6 (CKP, page 230, problem 7a)

Show that

$$K_{\nu}(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_{\nu}(z) = \frac{1}{z}.$$

Hint: Use recurrence relations for I and K derived from the ones for J.

Exercise 7 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \rightarrow +\infty$:

$$\begin{aligned} a) \quad I(x) &= \int_0^1 \sin \left[x \left(t + \frac{t^3}{6} - \sinh t \right) \right] dt, \\ b) \quad I(x) &= \int_{-1}^1 \sin [x(t - \sin t)] \sinh t dt. \end{aligned}$$

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.