Physics 503: Methods of Mathematical Physics

Read: CKP chapter 1

“CKP” refers to Carrier, Krook, and Pearson book.
Problems with stars are not for credit and will NOT be graded.

Homework 1

Exercise 1
Calculate real and imaginary parts of the following complex numbers:

\[ a) \ 2 + \sqrt{17}i \quad b) \ (\sqrt{2} - \sqrt{3}i)^2 \quad c) \ \frac{2+3i}{5-i} \]
\[ d) \ \left(\frac{1+i}{5}\right)^{17} \quad e) \ (2+5i)^{30} \quad f) \ \left(\frac{1+i}{5}\right)^{17} + (2 + 5i)^{30} \]

Exercise 2
Find \( \sin(3\theta) \) in terms of \( \sin \theta \) using de Moivre’s formula and identity \( \sin^2 \theta = 1 - \cos^2 \theta \).

*Exercise 3
Consider the sequence defined by recurrent relation and initial conditions.
\[
F_k = 2F_{k-1} - 2F_{k-2}, \\
F_0 = 1, \\
F_1 = 5.
\]
Write down the geometric sequence ansatz and find the roots of the corresponding quadratic equation. Write down the formula for \( F_k \) using initial conditions. Analyze the result using de Moivre’s formula. What is the value (order of magnitude) of \( F_{103} \)?

*Exercise 4 (CKP, page 5, problem 6)
Find the value of the following sum in a compact form
\[
1 + r \cos \theta + r^2 \cos 2\theta + \ldots + r^n \cos n\theta.
\]
Hint: Write it as a real part of a complex geometric sequence using de Moivre’s formula, sum it up, and find the real part of the result.
Exercise 5

Calculate real and imaginary parts of the principal value of the following complex numbers:

\[ a) \ln(1 + \sqrt{3}i) \quad b) \ln(-5) \]
\[ c) 2^{-i} \quad d) (1 - 3i)^{1/3} \]

Exercise 6

Find “all” multiple values (in arbitrary form) of the following expressions

\[ a) \ln(1 + \sqrt{3}i) \quad b) 1^{3/5} \quad c) (1 + \sqrt{3i})^{1/3} \]
\[ d) \left(z^{1/2}\right)^{1/3} \quad e) \left(z^{5/2}\right)^{2/5} \quad f) \ln(\ln i) \]

*Exercise 7

Show that the cross-ratio is an invariant of fractional transformation, i.e., that

\[
\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}
\]

when \( w = \frac{az + b}{cz + d} \).

Exercise 8 (CKP, page 19, problem 1)

Use the cross-ratio to obtain a mapping which transforms the upper-half \( z \) plane into the interior of the unit circle in the \( w \) plane. Sketch the \( w \) images of various points and curves in the \( z \) plane, and vice versa. (Hint: Replace \( z_1, w_1 \) by \( z, w \); set \( z_2, z_3, z_4 \) equal to \(-1, 0, 1\), etc. Or use point at infinity.)

*Exercise 9 (CKP, page 24, problem 13a)

Discuss the branch cut and Riemann-surface situation for the following function

\[ g(z) = \sqrt{1 + \sqrt{z}}. \]