Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 6-2 – 6-4, 5-1.
BO sections 6.4 – 6.7.

"**CKP**" refers to Carrier, Krook, and Pearson book.
"**BO**" refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 6

Exercise 1 (BO 6.26)

a) Obtain three terms of the asymptotic expansion of $I(x) = \int_0^{\pi/2} e^{-x\tan^2\theta}\, d\theta$ as $x \to \infty$.

b) Find the leading behavior of $I(x) = \int_0^{2\pi}(1+t^4)e^{x\cos t}\, dt$ as $x \to \infty$. Note that two maxima contribute to this leading behavior.

Exercise 2 (BO 6.28abc)

Find the leading behaviors of

$$\begin{aligned} a)\quad I(x) &= \int_0^{\pi/2} \sqrt{\sin t}\, e^{-x\sin^4 t}\, dt \quad \text{as } x \to \infty; \\ b)\quad I(x) &= \int_0^{1} \sqrt{t(1-t)}\, (t+a)^{-x}\, dt \quad \text{as } x \to +\infty \text{ with } a > 0; \\ c)\quad I(x) &= \int_0^{\pi/4} \sqrt{\tan t}\, e^{-xt^2}\, dt \quad \text{as } x \to +\infty. \end{aligned}$$

*Exercise 3

Find the leading behavior (both exponent and pre-exponential factor) of the integral $I(\omega) = \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(\cosh t)^{2/3}}\, dt$ as $\omega \to +\infty$.

Exercise 4 (BO 6.74)

Find three terms in the asymptotic behavior of $I(x) = \int_0^1 \ln(1+t)e^{ix\sin^2 t}\, dt$ as $x \to +\infty$.

Exercise 5 (BO 6.56abc)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$a) \quad I(x) = \int_0^1 e^{ixt^2} \cosh t^2 \, dt, \qquad b) \quad I(x) = \int_0^1 e^{ix(t-\tan t)} \, dt.$$

$$*c) \quad I(x) = \int_0^1 \cos(xt^4) \tan t \, dt,$$

Exercise 6 (BO 6.92b)

Find the leading behavior of the sum $S(x) = \sum_{k=0}^{\infty} \frac{1}{(k^2+x^2)^3}$ as $x \to +\infty$.

Exercise 7 (BO 6.93ac)

Find three terms in the asymptotic behavior as $n \to +\infty$ of the following sums:

$$a) \quad S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}, \qquad *b) \quad S_n = \sum_{k=1}^{n} \frac{\sin k}{k}.$$

Exercise 8 (FS 81.12)

Express the following integrals in terms of the Γ-function:

$$a) \quad I = \int_0^{\pi/2} \sin^{2p} \phi \cos^{2q} \phi \, d\phi,$$

$$*b) \quad I = \int_0^{\pi/2} \tan^p \phi \, d\phi,$$

$$*c) \quad I = \int_0^1 \frac{dx}{\sqrt{1-x^4}},$$

$$d) \quad I = \int_0^{+\infty} \frac{x^n \, dx}{(a+bx^m)^p},$$

where $a > 0$, $b > 0$, $mp > n + 1$.

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.