Physics 503: Methods of Mathematical Physics

Read: CKP sections 5-1, 5-5, 6-4.
       BO sections 6.5.

“CKP” refers to Carrier, Krook, and Pearson book.
“BO” refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 7

*Exercise 1
Evaluate the integral \( \int_0^1 \psi(x) \sin(2\pi nx) \, dx \) for \( n = 0, 1, 2, \ldots \). Here \( \psi(x) \) is the digamma function.

Exercise 2
Evaluate the integral (exactly)
\[
\int \frac{d^2p}{(2\pi)^2} \frac{e^{i\vec{p} \cdot \vec{r}}}{p^2 + m^2},
\]
where the integral is taken over two-dimensional \( p \)-plane, \( \vec{p} \cdot \vec{r} = p_1 x + p_2 y \), \( p = \sqrt{p_1^2 + p_2^2} \), and \( m \) is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as \( r \to \infty \). Find the leading asymptotic behavior of the result as \( r \to 0 \). Can you obtain the latter not from the exact result but directly from the integral?

Exercise 3
Find three terms of an asymptotic expansion of Macdonald’s function \( K_0(x) \) as \( x \to \infty \) starting from the integral representation
\[
K_0(x) = \int_0^\infty e^{-x \cosh t} \, dt.
\]

*Exercise 4
Calculate the following integral (exactly)
\[
\int_0^\infty e^{-ax} J_n(bx) \, dx.
\]
Hint: You can use the generating function for Bessel functions. First calculate the integral over $x$ and then find the coefficient in Laureant series in $t$ using the closed contour integral in $t$-plane.

**Exercise 5**

Calculate the following integral (exactly)

$$\int_0^\infty \frac{e^{-px}}{\sqrt{x(x+a)}} \, dx.$$

Here $a > 0$, $p > 0$.

*Hint:* The answer is given in terms of Macdonald’s function.

**Exercise 6 (CKP, page 230, problem 7a)**

Show that

$$K_\nu(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_\nu(z) = \frac{1}{z}.$$

*Hint:* Use recurrence relations for $I$ and $K$ derived from the ones for $J$.

**Exercise 7 (BO 6.56de)**

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

- a) $I(x) = \int_0^1 \sin \left[ x \left( t + \frac{t^3}{6} - \sinh t \right) \right] \, dt$,
- b) $I(x) = \int_{-1}^1 \sin \left[ x(t - \sin t) \right] \sinh t \, dt$.

The problems referred to as FS are taken from the book: