Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 5-1, 5-5, 6-4.
BO sections 6.5.

"**CKP**" refers to Carrier, Krook, and Pearson book.
"**BO**" refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 7

*Exercise 1

Evaluate the integral $\int_0^1 \psi(x)\sin(2\pi n x)\,dx$ for $n = 0, 1, 2, \ldots$. Here $\psi(x)$ is the digamma function.

Exercise 2

Evaluate the integral (exactly)

$$\int \frac{d^2p}{(2\pi)^2}\,\frac{e^{i\vec{p}\cdot\vec{r}}}{p^2+m^2},$$

where the integral is taken over two-dimensional p-plane, $\vec{p}\cdot\vec{r} = p_1 x + p_2 y$, $p = \sqrt{p_1^2+p_2^2}$, and m is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as $r \to \infty$. Find the leading asymptotic behavior of the result as $r \to 0$. Can you obtain the latter not from the exact result but directly from the integral?

Exercise 3

Find three terms of an asymptotic expansion of Macdonald's function $K_0(x)$ as $x \to \infty$ starting from the integral representation

$$K_0(x) = \int_0^\infty e^{-x\cosh t}\,dt.$$

*Exercise 4

Calculate the following integral (exactly)

$$\int_0^\infty e^{-ax} J_n(bx)\,dx.$$

Hint: You can use the generating function for Bessel functions. First calculate the integral over x and then find the coefficient in Laureant series in t using the closed contour integral in t-plane.

Exercise 5

Calculate the following integral (exactly)

$$\int_0^\infty \frac{e^{-px}dx}{\sqrt{x(x+a)}}.$$

Here $a > 0$, $p > 0$.

Hint: The answer is given in terms of Macdonald's function.

Exercise 6 (CKP, page 230, problem 7a)

Show that

$$K_\nu(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_\nu(z) = \frac{1}{z}.$$

Hint: Use recurrence relations for I and K derived from the ones for J.

Exercise 7 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$\begin{aligned} a) \quad I(x) &= \int_0^1 \sin\left[x\left(t + \frac{t^3}{6} - \sinh t\right)\right] dt, \\ b) \quad I(x) &= \int_{-1}^1 \sin\left[x(t - \sin t)\right] \sinh t \, dt. \end{aligned}$$

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.