12/07/2006. **Due:** Thursday, December 14, 2006 (in class).

Physics 503: Methods of Mathematical Physics

Read: CKP sections 5-1, 5-5, 6-4.
BO sections 6.5.

"**CKP**" refers to Carrier, Krook, and Pearson book.
"**BO**" refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 7

Exercise 1

Calculate the following integral (exactly)

$$\int_0^\infty e^{-ax} J_n(bx)\, dx.$$

Hint: You can use the generating function for Bessel functions. First calculate the integral over x and then find the coefficient in Laureant series in t using the closed contour integral in t-plane.

*Exercise 2

Calculate the following integral (exactly)

$$\int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}}.$$

Here $a > 0$, $p > 0$.

Hint: The answer is given in terms of Macdonald's function.

*Exercise 3 (CKP, page 230, problem 7a)

Show that

$$K_\nu(z) I_{\nu+1}(z) + K_{\nu+1}(z) I_\nu(z) = \frac{1}{z}.$$

Hint: Use recurrence relations for I and K derived from the ones for J.

Exercise 4 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$
\begin{aligned}
a) \quad I(x) &= \int_0^1 \sin\left[x\left(t + \frac{t^3}{6} - \sinh t\right)\right] dt, \\
b) \quad I(x) &= \int_{-1}^1 \sin\left[x(t - \sin t)\right] \sinh t \, dt.
\end{aligned}
$$

*Exercise 5

Find the nature of each singularity (including the point at infinity) of each of the following functions.

$$a) \quad \frac{\sqrt{z(1-z)}}{(e^z-3)^2}, \qquad b) \quad z^2 e^{\frac{1}{z}}.$$

Evaluate the residues at each isolated singularity. Always include the point at ∞ in your considerations.

Exercise 6

Evaluate the following integral

$$I = \int_0^\infty \frac{\sin ax}{x(1+x^4)} \, dx.$$

Exercise 7

Find the flow of a fluid (complex potential) in the sector $0 < \arg z < \pi/3$ produced by a source of strength Q concentrated at the point $z_0 = ae^{i\pi/6}$.

Exercise 8

Find the leading behavior of the following integral as $x \to +\infty$.

$$\int_{-1}^1 \cos\left[x\left(1 - \frac{t^2}{2} - \cos t\right)\right] (\cosh t - 1) \, dt.$$

*Exercise 9

Nematic is a liquid crystal characterized by an order parameter which is the unit three-component vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$ with an additional condition $\vec{n} \sim -\vec{n}$. The latter means that two unit vectors which are opposite to each other describe the same state.

What types of topological defects and textures are allowed for three-dimensional nematic? What about two-dimensional one?

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.