Physics 503: Methods of Mathematical Physics

Read: CKP chapter 1

“CKP” refers to Carrier, Krook, and Pearson book.
Problems with stars are not for credit and will NOT be graded.

Homework 1

Exercise 1

Calculate real and imaginary parts of the following complex numbers:

\[ a) \quad 2 + \sqrt{17}i \quad b) \quad (\sqrt{2} - \sqrt{3}i)^2 \quad c) \quad \frac{2+3i}{5-i} \]
\[ d) \quad \left(\frac{1+i}{5}\right)^{17} \quad e) \quad (2+5i)^{30} \quad f) \quad \left(\frac{1+i}{5}\right)^{17} + (2+5i)^{30} \]

Exercise 2

Find \( \sin(3\theta) \) in terms of \( \sin \theta \) using de Moivre’s formula and identity \( \sin^2 \theta = 1 - \cos^2 \theta \).

Exercise 3

Consider the sequence defined by recurrent relation and initial conditions.

\[ F_k = 2F_{k-1} - 2F_{k-2}, \]
\[ F_0 = 1, \]
\[ F_1 = 5. \]

Write down the geometric sequence ansatz and find the roots of the corresponding quadratic equation. Write down the formula for \( F_k \) using initial conditions. Analyze the result using de Moivre’s formula. What is the value (order of magnitude) of \( F_{103} \)?

*Exercise 4 (CKP, page 5, problem 6)

Find the value of the following sum in a compact form

\[ 1 + r \cos \theta + r^2 \cos 2\theta + \ldots + r^n \cos n\theta. \]

Hint: Write it as a real part of a complex geometric sequence using de Moivre’s formula, sum it up, and find the real part of the result.
*Exercise 5

Calculate real and imaginary parts of the principal value of the following complex numbers:

\[ \begin{align*}
  a) & \quad \ln(1 + \sqrt{3}i) \\
  b) & \quad \ln(-5) \\
  c) & \quad 2^{-i} \\
  d) & \quad (1 - 3i)^{1/3}
\end{align*} \]

Exercise 6

Find “all” multiple values (in arbitrary form) of the following expressions

\[ \begin{align*}
  a) & \quad \ln(1 + \sqrt{3}i) \\
  b) & \quad 1^{3/5} \\
  c) & \quad (1 + \sqrt{3}i)^{1/3} \\
  d) & \quad \left(z^{1/2}\right)^{1/3} \\
  e) & \quad \left(z^{5/2}\right)^{2/5} \\
  f) & \quad \ln(\ln i)
\end{align*} \]

*Exercise 7

Show that the cross-ratio is an invariant of fractional transformation, i.e., that

\[ \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \]

when \( w = \frac{az + b}{cz + d} \).

Exercise 8 (CKP, page 19, problem 1)

Use the cross-ratio to obtain a mapping which transforms the upper-half \( z \) plane into the interior of the unit circle in the \( w \) plane. Sketch the \( w \) images of various points and curves in the \( z \) plane, and vice versa. (Hint: Replace \( z_1, w_1 \) by \( z, w \); set \( z_2, z_3, z_4 \) equal to \(-1, 0, 1\), etc. Or use point at infinity.)

*Exercise 9 (CKP, page 24, problem 13a)

Discuss the branch cut and Riemann-surface structure for the following function

\[ g(z) = \sqrt{1 + \sqrt{z}}. \]