Physics 503: Methods of Mathematical Physics

Read: CKP sections 6-2 – 6-4, 5-1.  
BO sections 6.4 – 6.7.

“CKP” refers to Carrier, Krook, and Pearson book.  
“BO” refers to Bender and Orszag book.  
Problems with stars are not for credit and will NOT be graded.

Homework 6

*Exercise 1
Find the leading behavior (both exponent and pre-exponential factor) of the integral
\[ I(\omega) = \int_{-\infty}^{+\infty} e^{-\frac{i\omega t}{\cosh t}} \frac{dt}{t^3} \] as \( \omega \to +\infty \).

Exercise 2 (BO 6.74)
Find three terms in the asymptotic behavior of 
\[ I(x) = \int_0^1 \ln(1 + t)e^{ix^2} \sin^2 t \ dt \] as \( x \to +\infty \).

Exercise 3 (BO 6.56abc)
Use the method of stationary phase to find the leading behavior of the following integrals as \( x \to +\infty \):

a) \[ I(x) = \int_0^1 e^{ix^2} \cosh t^2 dt, \]
b) \[ I(x) = \int_0^1 e^{ix(t - \tan t)} dt. \]
*c) \[ I(x) = \int_0^1 \cos(xt^2) \tan \frac{t}{x} dt, \]

Exercise 4 (BO 6.92b)
Find the leading behavior of the sum 
\[ S(x) = \sum_{k=0}^\infty \frac{1}{(k^2 + x^2)x} \] as \( x \to +\infty \).

Exercise 5 (BO 6.93ac)
Find three terms in the asymptotic behavior as \( n \to +\infty \) of the following sums:

a) \[ S_n = \sum_{k=1}^n \frac{(-1)^k}{k}, \]
*b) \[ S_n = \sum_{k=1}^n \frac{\sin k}{k}. \]
Exercise 6 (FS 81.12)
Express the following integrals in terms of the Γ-function:

\[ a) \quad I = \int_{0}^{\pi/2} \sin^{2p} \phi \cos^{2q} \phi \, d\phi, \]

\[ *b) \quad I = \int_{0}^{\pi/2} \tan^{p} \phi \, d\phi, \]

\[ *c) \quad I = \int_{1}^{\infty} \frac{dx}{\sqrt{1-x^2}}, \]

\[ d) \quad I = \int_{0}^{+\infty} \frac{x^n \, dx}{(a + bx^m)^p}, \]

where \( a > 0, b > 0, mp > n + 1. \)

*Exercise 7
Evaluate the integral \( \int_{0}^{1} \psi(x) \sin(2\pi nx) \, dx \) for \( n = 0, 1, 2, \ldots \). Here \( \psi(x) \) is the digamma function.

Exercise 8
Evaluate the integral (exactly)

\[ \int \frac{d^2p}{(2\pi)^2 \, p^2 + m^2} \, e^{\i \vec{p} \cdot \vec{r}}, \]

where the integral is taken over two-dimensional \( p \)-plane, \( \vec{p} \cdot \vec{r} = p_1 x + p_2 y, \) \( p = \sqrt{p_1^2 + p_2^2} \), and \( m \) is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as \( r \to \infty \). Find the leading asymptotic behavior of the result as \( r \to 0 \). Can you obtain the latter not from the exact result but directly from the integral?

Exercise 9
Find three terms of an asymptotic expansion of Macdonald’s function \( K_0(x) \) as \( x \to \infty \) starting from the integral representation

\[ K_0(x) = \int_{0}^{\infty} e^{-x \cosh t} \, dt. \]

The problems referred to as FS are taken from the book: