Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 6-2 – 6-4, 5-1.
BO sections 6.4 – 6.7.

"**CKP**" refers to Carrier, Krook, and Pearson book.
"**BO**" refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 6

*Exercise 1

Find the leading behavior (both exponent and pre-exponential factor) of the integral $I(\omega) = \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(\cosh t)^{2/3}}\, dt$ as $\omega \to +\infty$.

Exercise 2 (BO 6.74)

Find three terms in the asymptotic behavior of $I(x) = \int_0^1 \ln(1+t) e^{ix\sin^2 t}\, dt$ as $x \to +\infty$.

Exercise 3 (BO 6.56abc)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

$$a)\quad I(x) = \int_0^1 e^{ixt^2} \cosh t^2\, dt, \qquad b)\quad I(x) = \int_0^1 e^{ix(t-\tan t)}\, dt.$$

$$*c)\quad I(x) = \int_0^1 \cos(xt^4) \tan t\, dt,$$

Exercise 4 (BO 6.92b)

Find the leading behavior of the sum $S(x) = \sum_{k=0}^{\infty} \frac{1}{(k^2+x^2)^3}$ as $x \to +\infty$.

Exercise 5 (BO 6.93ac)

Find three terms in the asymptotic behavior as $n \to +\infty$ of the following sums:

$$a)\quad S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}, \qquad *b)\quad S_n = \sum_{k=1}^{n} \frac{\sin k}{k}.$$

Exercise 6 (FS 81.12)

Express the following integrals in terms of the Γ-function:

$$\begin{aligned}
a)\ I &= \int_0^{\pi/2} \sin^{2p}\phi \cos^{2q}\phi \, d\phi, \\
{}^*b)\ I &= \int_0^{\pi/2} \tan^{p}\phi \, d\phi, \\
{}^*c)\ I &= \int_0^{1} \frac{dx}{\sqrt{1-x^4}}, \\
d)\ I &= \int_0^{+\infty} \frac{x^n \, dx}{(a+bx^m)^p},
\end{aligned}$$

where $a > 0$, $b > 0$, $mp > n+1$.

*Exercise 7

Evaluate the integral $\int_0^1 \psi(x)\sin(2\pi n x)\,dx$ for $n = 0, 1, 2, \ldots$. Here $\psi(x)$ is the digamma function.

Exercise 8

Evaluate the integral (exactly)

$$\int \frac{d^2p}{(2\pi)^2}\,\frac{e^{i\vec{p}\cdot\vec{r}}}{p^2+m^2},$$

where the integral is taken over two-dimensional p-plane, $\vec{p}\cdot\vec{r} = p_1 x + p_2 y$, $p = \sqrt{p_1^2+p_2^2}$, and m is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as $r \to \infty$. Find the leading asymptotic behavior of the result as $r \to 0$. Can you obtain the latter not from the exact result but directly from the integral?

Exercise 9

Find three terms of an asymptotic expansion of Macdonald's function $K_0(x)$ as $x \to \infty$ starting from the integral representation

$$K_0(x) = \int_0^{\infty} e^{-x\cosh t}\,dt.$$

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.