Physics 503: Methods of Mathematical Physics

Read: CKP sections 5-1, 5-5, 6-4.
        BO sections 6.5.

“CKP” refers to Carrier, Krook, and Pearson book.
“BO” refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 7

Exercise 1
Calculate the following integral (exactly)
\[ \int_{0}^{\infty} e^{-ax} J_n(bx) \, dx. \]

*Hint:* You can use the generating function for Bessel functions. First calculate the integral over \( x \) and then find the coefficient in Laurent series in \( t \) using the closed contour integral in \( t \)-plane.

*Exercise 2
Calculate the following integral (exactly)
\[ \int_{0}^{\infty} \frac{e^{-px} \, dx}{\sqrt{x(x+a)}}. \]

Here \( a > 0, \ p > 0. \)

*Hint:* The answer is given in terms of Macdonald’s function.

*Exercise 3 (CKP, page 230, problem 7a)
Show that
\[ K_\nu(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_\nu(z) = \frac{1}{z}. \]

*Hint:* Use recurrence relations for I and K derived from the ones for J.
Exercise 4 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

\[ a) \quad I(x) = \int_0^1 \sin \left[ x \left( t + \frac{t^3}{6} - \sinh t \right) \right] \, dt, \]

\[ b) \quad I(x) = \int_{-1}^1 \sin [x(t - \sin t)] \sinh t \, dt. \]

*Exercise 5

Find the nature of each singularity (including the point at infinity) of each of the following functions.

\[ a) \quad \frac{\sqrt{z(1-z)}}{(e^z - 3)^2}, \quad b) \quad z^2 e^{\frac{z}{2}}. \]

Evaluate the residues at each isolated singularity. Always include the point at $\infty$ in your considerations.

Exercise 6

Evaluate the following integral

\[ I = \int_0^\infty \frac{\sin ax}{x(1 + x^4)} \, dx. \]

Exercise 7

Find the flow of a fluid (complex potential) in the sector $0 < \arg z < \pi/3$ produced by a source of strength $Q$ concentrated at the point $z_0 = ae^{i\pi/6}$.

Exercise 8

Find the leading behavior of the following integral as $x \to +\infty$.

\[ \int_{-1}^1 \cos \left[ x \left( 1 - \frac{t^2}{2} - \cos t \right) \right] (\cosh t - 1) \, dt. \]

*Exercise 9

Nematic is a liquid crystal characterized by an order parameter which is the unit three-component vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$ with an additional condition $\vec{n} \sim -\vec{n}$. The latter means that two unit vectors which are opposite to each other describe the same state.

What types of topological defects and textures are allowed for three-dimensional nematic? What about two-dimensional one?
**Exercise 10**

The Poisson’s summation formula is

\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \tilde{F}(2\pi m), \]

where the Fourier transform is defined as

\[ \tilde{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t} dt. \]

a) Prove the Poisson’s formula.
b) Find the leading behavior of the following sum as \( A \to +0 \) using the Poisson’s formula.

\[ \sum_{n \in \mathbb{Z}} e^{-\frac{A}{2}(n-\bar{n})^2}. \]

The problems referred to as FS are taken from the book: