Physics 503: Methods of Mathematical Physics

Read: CKP chapter 2, sections 2-1 — 2-7.

“CKP” refers to Carrier, Krook, and Pearson book. Problems with stars are not for credit and will NOT be graded.

Homework 2

*Exercise 1 (CKP, page 29, problem 2)

Verify the Cauchy-Riemann equations for \((1 - z^2)^{1/2}\). At what points this function has singularities?

Exercise 2 (CKP, page 29, problem 2)

Prove in an easy way that \((x^2 + y^2)^{1/4} \cos\left(\frac{1}{2} \arctan \frac{y}{x}\right)\) is harmonic.

*Exercise 3 (CKP, page 30, problem 7)

If \(u\) and \(v\) are expressed in terms of polar coordinates \((r, \theta)\), show that the Cauchy-Riemann equations can be written

\[
\frac{u_r}{r} = \frac{v_\theta}{r}, \quad \frac{1}{r} u_\theta = -v_r.
\]

Exercise 4 (CKP, page 36, problem 3)

Show in an easy way that the integral of each of the following expressions around the circle \(|z| = 1/2\) vanishes:

\[
a) \quad \frac{z^{2+1}}{z^2 + z + 1}, \quad b) \quad e^{z^2} \ln(1 + z), \quad c) \quad \arcsin z.
\]

Exercise 5 (CKP, page 40, problem 1)

Use Cauchy’s integral formula to evaluate the integral around the unit circle \((|z| = 1)\) of

\[
a) \quad \frac{\sin z}{2z + 1}, \quad b) \quad \frac{\ln(z+2)}{z+2}, \quad c) \quad \frac{z^3 + \arcsinh(z/2)}{z^2 + z + 1}.
\]
Exercise 6
Find the principal value of the integral \( \int_C \frac{\sin z}{z^2} \, dz \) where counterclockwise contour \( C \) is a square \( ABDF \) with \( A = 0, B = 2\pi, D = 2\pi(1 + i), \) and \( F = 2\pi i \).

*Exercise 7 (CKP, page 43, problem 1)
Find the maximum for \( |z| \leq 1 \) of functions
\[ a) \quad |z^2 + 2z + i|, \quad b) \quad |\sin(z)|, \quad c) \quad |\arcsin \frac{z}{2}|. \]

Exercise 8
Show that the Cauchy-Riemann equations for modulus and argument of function \( f(z) = |f|e^{i\theta} \) can be written in the form
\[(\ln |f|)_x = \theta_y, \quad (\ln |f|)_y = -\theta_x.\]

*Exercise 9 (CKP, page 56, problem 3)
Expand in powers of \( z \) the function \( \sin(z + 1/z) \) in whatever annual region is closest to the origin. Express the coefficients as simple (purely real) trigonometric integrals.

Exercise 10
Discuss the character of the singularities of functions
\[ a) \quad \frac{1}{(z^2 + 2)^2(z - 1)}, \quad b) \quad \cot^2(z), \quad c) \quad \frac{1}{\sin z^2}, \quad d) \quad \frac{1}{(z^2 - 1)^{1/2} + z + 1}. \]
Include the point at \( \infty \) in your considerations.

Exercise 11
Discuss the character of the singularities of the following functions \((a > 0)\)
\[ a) \quad \frac{1}{z^2 + a^2}, \quad b) \quad \frac{z^2}{z^2 + a^2}, \quad c) \quad \frac{\sin(1/z)}{z^2 + a^2}, \quad d) \quad \frac{ze^{iz}}{z^2 - a^2}. \]
Always include the point at \( \infty \) in your considerations. Evaluate the residues at isolated singularities (and at \( \infty \) if it is possible).

*Exercise 12
Same as in Ex.11.
\[ a) \quad \frac{z^{-k}}{z^{k+1}}, \quad 0 < k < 1, \quad b) \quad \frac{z^{-3}}{z\sqrt{z^2 - a^2}}, \quad c) \quad \frac{\ln z}{\sqrt{z^2 + a^2}}, \quad d) \quad \frac{\cos az}{(z^2 + 1)^2}, \quad e) \quad \tan z. \]