9/16/2008. **Due:** Thursday, October 2, 2008 (in class).

Physics 503: Methods of Mathematical Physics

Read: CKP chapter 2, sections 2-1 — 2-7.

"**CKP**" refers to Carrier, Krook, and Pearson book.
Problems with stars are not for credit and will NOT be graded.

Homework 2

*Exercise 1 (CKP, page 29, problem 2)

Verify the Cauchy-Riemann equations for $(1-z^2)^{1/2}$. At what points this function has singularities?

Exercise 2 (CKP, page 29, problem 2)

Prove in an easy way that $(x^2+y^2)^{1/4}\cos\left(\frac{1}{2}\arctan\frac{y}{x}\right)$ is harmonic.

*Exercise 3 (CKP, page 30, problem 7)

If u and v are expressed in terms of polar coordinates (r,θ), show that the Cauchy-Riemann equations can be written

$$u_r = \frac{1}{r}v_\theta, \qquad \frac{1}{r}u_\theta = -v_r.$$

Exercise 4 (CKP, page 36, problem 3)

Show in an easy way that the integral of each of the following expressions around the circle $|z| = 1/2$ vanishes:

$$a)\quad \frac{z+1}{z^2+z+1}, \qquad b)\quad e^{z^2}\ln(1+z), \qquad c)\quad \arcsin z.$$

Exercise 5 (CKP, page 40, problem 1)

Use Cauchy's integral formula to evaluate the integral around the unit circle ($|z| = 1$) of

$$a)\quad \frac{\sin z}{2z+i}, \qquad b)\quad \frac{\ln(z+2)}{z+2}, \qquad c)\quad \frac{z^3+\text{arcsinh}\,(z/2)}{z^2+iz+i}.$$

Exercise 6

Find the principal value of the integral $\int_C \frac{\sin z}{z^2}\, dz$ where counterclockwise contour C is a square $ABDF$ with $A = 0$, $B = 2\pi$, $D = 2\pi(1+i)$, and $F = 2\pi i$.

*Exercise 7 (CKP, page 43, problem 1)

Find the maximum for $|z| \leq 1$ of functions

$$a)\quad |z^2 + 2z + i|, \qquad b)\quad |\sin(z)|, \qquad c)\quad |\arcsin \tfrac{z}{2}|.$$

Exercise 8

Show that the Cauchy-Riemann equations for modulus and argument of function $f(z) = |f|e^{i\theta}$ can be written in the form

$$(\ln|f|)_x = \theta_y, \qquad (\ln|f|)_y = -\theta_x.$$

*Exercise 9 (CKP, page 56, problem 3)

Expand in powers of z the function $\sin(z + 1/z)$ in whatever annual region is closest to the origin. Express the coefficients as simple (purely real) trigonometric integrals.

Exercise 10

Discuss the character of the singularities of functions

$$a)\quad \frac{1}{(z^2+2)^2(z-i)}, \qquad b)\quad \cot^2(z), \qquad c)\quad \frac{1}{\sin z^2}, \qquad d)\quad \frac{1}{(z^2-1)^{1/2}+z+i}.$$

Include the point at ∞ in your considerations.

Exercise 11

Discuss the character of the singularities of the following functions $(a > 0)$

$$a)\quad \frac{1}{z^2+a^2}, \qquad b)\quad \frac{z^2}{z^2+a^2}, \qquad c)\quad \frac{\sin(1/z)}{z^2+a^2}, \qquad d)\quad \frac{ze^{iz}}{z^2-a^2}.$$

Always include the point at ∞ in your considerations. Evaluate the residues at isolated singularities (and at ∞ if it is possible).

*Exercise 12

Same as in Ex.11.

$$a)\quad \frac{z^{-k}}{z+1},\quad 0 < k < 1, \qquad b)\quad \frac{z-3}{z\sqrt{z^2-a^2}}, \qquad c)\quad \frac{\ln z}{\sqrt{z^2+a^2}},$$

$$d)\quad \frac{\cos az}{(z^2+1)^2}, \qquad e)\quad \tan z.$$