Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 6-1 – 6-4.
BO sections 3.8, 6.1 – 6.7.

"**CKP**" refers to Carrier, Krook, and Pearson book.
"**BO**" refers to Bender and Orszag book.
Problems with stars are not for credit and will NOT be graded.

Homework 5

Exercise 1

Find the conformal mapping of the upper half-plane onto itself which maps the points $\infty, 0, 1$ onto $0, 1, \infty$ respectively.

Exercise 2

Find the complex potential and the stream lines for the plane flow of a liquid in the first quadrant when there is a source of strength Q at $z = 1 + i$ and a sink of equal strength at $z = 0$.

Exercise 3 (CKP, page 244, problem 4a)

Use integration by parts to obtain asymptotic expansions, valid for large x, for the integral

$$I(x) = \int_0^1 (\cos t + t^2) e^{ixt}\, dt.$$

Exercise 4 (CKP, page 244, problem 4b)

Use integration by parts to obtain asymptotic expansions, valid for large x, for the integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{\sqrt{t}}\, dt.$$

Hint: Write $\int_0^1 = \int_0^\infty - \int_1^\infty$.

Exercise 5 (CKP, page 254, problem 5ac)

Obtain the first few terms of the asymptotic expansions, as $x \to \infty$, of

a) $I(x) = \int_0^\pi \frac{e^{-xt^2}}{t^{1/3}} \cos t\, dt,$ *b*) $I(x) = \int_0^1 t^x \sin^2 t\, dt.$

*Exercise 6 (BO 6.26)

a) Obtain three terms of the asymptotic expansion of $I(x) = \int_0^{\pi/2} e^{-x\tan^2\theta}\, d\theta$ as $x \to \infty$.

b) Find the leading behavior of $I(x) = \int_0^{2\pi} (1 + t^4) e^{x\cos t}\, dt$ as $x \to \infty$. Note that two maxima contribute to this leading behavior.

Exercise 7 (BO 6.28abc)

Find the leading behaviors of

$$\begin{aligned} a)\quad I(x) &= \int_0^{\pi/2} \sqrt{\sin t}\, e^{-x\sin^4 t}\, dt \quad \text{as } x \to \infty; \\ b)\quad I(x) &= \int_0^{1} \sqrt{t(1-t)}\, (t+a)^{-x}\, dt \quad \text{as } x \to +\infty \text{ with } a>0; \\ c)\quad I(x) &= \int_0^{\pi/4} \sqrt{\tan t}\, e^{-xt^2}\, dt \quad \text{as } x \to +\infty. \end{aligned}$$

*Exercise 8 (CKP, page 126, problem 14)

The function $g(z;z_0)$ defined on the domain $z \in D$ so that $z_0 \in D$ is an arbitrary point of the domain is called the Green's function of the Laplace equation on this domain if (i) $g(z;z_0)$ is harmonic for all $z \in D$ except for $z = z_0$, (ii) $g(z \in \partial D; z_0) = 0$, (iii) $\frac{g(z;z_0)}{\ln|z-z_0|} \to 1$ as $z \to z_0$.

a) Show that if $\zeta = F(z;z_0)$ maps conformally the simply connected domain D of the z-plane onto the unit circle $|\zeta| < 1$ so that z_0 maps to $\zeta = 0$ then

$$g(z;z_0) = \ln|F(z;z_0)|.$$

b) Show that if $\zeta = F(z)$ maps conformally the simply connected domain D of the z-plane onto the unit circle $|\zeta| < 1$ then

$$g(z;z_0) = -\ln\left|\frac{1 - F^*(z_0)F(z)}{F(z) - F(z_0)}\right|.$$

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.