

## Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 6-2 – 6-4, 5-1.

**BO** sections 6.4 – 6.7.

“**CKP**” refers to Carrier, Krook, and Pearson book.

“**BO**” refers to Bender and Orszag book.

Problems with stars are not for credit and will NOT be graded.

### Homework 6

#### \*Exercise 1

Find the leading behavior (both exponent and pre-exponential factor) of the integral

$$I(\omega) = \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(\cosh t)^{2/3}} dt \text{ as } \omega \rightarrow +\infty.$$

#### Exercise 2 (BO 6.74)

Find three terms in the asymptotic behavior of  $I(x) = \int_0^1 \ln(1+t)e^{ix \sin^2 t} dt$  as  $x \rightarrow +\infty$ .

#### Exercise 3 (BO 6.56abc)

Use the method of stationary phase to find the leading behavior of the following integrals as  $x \rightarrow +\infty$ :

$$\begin{aligned} a) \quad I(x) &= \int_0^1 e^{ixt^2} \cosh t^2 dt, & b) \quad I(x) &= \int_0^1 e^{ix(t-\tan t)} dt. \\ *c) \quad I(x) &= \int_0^1 \cos(xt^4) \tan t dt, \end{aligned}$$

#### Exercise 4 (BO 6.92b)

Find the leading behavior of the sum  $S(x) = \sum_{k=0}^{\infty} \frac{1}{(k^2+x^2)^3}$  as  $x \rightarrow +\infty$ .

#### Exercise 5 (BO 6.93ac)

Find three terms in the asymptotic behavior as  $n \rightarrow +\infty$  of the following sums:

$$a) \quad S_n = \sum_{k=1}^n \frac{(-1)^k}{k}, \quad *b) \quad S_n = \sum_{k=1}^n \frac{\sin k}{k}.$$

## Exercise 6 (FS 81.12)

Express the following integrals in terms of the  $\Gamma$ -function:

$$a) \quad I = \int_0^{\pi/2} \sin^{2p} \phi \cos^{2q} \phi \, d\phi,$$

$$*b) \quad I = \int_0^{\pi/2} \tan^p \phi \, d\phi,$$

$$*c) \quad I = \int_0^1 \frac{dx}{\sqrt{1-x^4}},$$

$$d) \quad I = \int_0^{+\infty} \frac{x^n dx}{(a + bx^m)^p},$$

where  $a > 0$ ,  $b > 0$ ,  $mp > n + 1$ .

## \*Exercise 7

Evaluate the integral  $\int_0^1 \psi(x) \sin(2\pi nx) dx$  for  $n = 0, 1, 2, \dots$ . Here  $\psi(x)$  is the digamma function.

## Exercise 8

Evaluate the integral (exactly)

$$\int \frac{d^2p}{(2\pi)^2} \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2 + m^2},$$

where the integral is taken over two-dimensional  $p$ -plane,  $\vec{p}\cdot\vec{r} = p_1x + p_2y$ ,  $p = \sqrt{p_1^2 + p_2^2}$ , and  $m$  is a real positive constant. Find the leading asymptotic behavior (both exponent and pre-exponential factor) of the result as  $r \rightarrow \infty$ . Find the leading asymptotic behavior of the result as  $r \rightarrow 0$ . Can you obtain the latter not from the exact result but directly from the integral?

## Exercise 9

Find three terms of an asymptotic expansion of Macdonald's function  $K_0(x)$  as  $x \rightarrow \infty$  starting from the integral representation

$$K_0(x) = \int_0^\infty e^{-x \cosh t} dt.$$

---

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.