

Physics 503: Methods of Mathematical Physics

Read: **CKP** sections 5-1, 5-5, 6-4.

BO sections 6.5.

“**CKP**” refers to Carrier, Krook, and Pearson book.

“**BO**” refers to Bender and Orszag book.

Problems with stars are not for credit and will NOT be graded.

Homework 7

Exercise 1

Calculate the following integral (exactly)

$$\int_0^{\infty} e^{-ax} J_n(bx) dx.$$

Hint: You can use the generating function for Bessel functions. First calculate the integral over x and then find the coefficient in Laurent series in t using the closed contour integral in t -plane.

*Exercise 2

Calculate the following integral (exactly)

$$\int_0^{\infty} \frac{e^{-px} dx}{\sqrt{x(x+a)}}.$$

Here $a > 0$, $p > 0$.

Hint: The answer is given in terms of Macdonald's function.

*Exercise 3 (CKP, page 230, problem 7a)

Show that

$$K_{\nu}(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_{\nu}(z) = \frac{1}{z}.$$

Hint: Use recurrence relations for I and K derived from the ones for J.

Exercise 4 (BO 6.56de)

Use the method of stationary phase to find the leading behavior of the following integrals as $x \rightarrow +\infty$:

$$a) \quad I(x) = \int_0^1 \sin \left[x \left(t + \frac{t^3}{6} - \sinh t \right) \right] dt,$$

$$b) \quad I(x) = \int_{-1}^1 \sin [x(t - \sin t)] \sinh t \, dt.$$

*Exercise 5

Find the nature of each singularity (including the point at infinity) of each of the following functions.

$$a) \quad \frac{\sqrt{z(1-z)}}{(e^z-3)^2}, \quad b) \quad z^2 e^{\frac{1}{z}}.$$

Evaluate the residues at each isolated singularity. Always include the point at ∞ in your considerations.

Exercise 6

Evaluate the following integral

$$I = \int_0^\infty \frac{\sin ax}{x(1+x^4)} dx.$$

Exercise 7

Find the flow of a fluid (complex potential) in the sector $0 < \arg z < \pi/3$ produced by a source of strength Q concentrated at the point $z_0 = ae^{i\pi/6}$.

Exercise 8

Find the leading behavior of the following integral as $x \rightarrow +\infty$.

$$\int_{-1}^1 \cos \left[x \left(1 - \frac{t^2}{2} - \cos t \right) \right] (\cosh t - 1) \, dt.$$

*Exercise 9

The Poisson's summation formula is

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \tilde{F}(2\pi m),$$

where the Fourier transform is defined as

$$\tilde{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t} dt.$$

- a) Prove the Poisson's formula.
- b) Find the leading behavior of the following sum as $A \rightarrow +0$ using the Poisson's formula.

$$\sum_{n \in \mathbb{Z}} e^{-\frac{A}{2}(n-\bar{n})^2}.$$

The problems referred to as FS are taken from the book:

B. A. Fuchs and B. V. Shabat, *Functions of a complex variable and some of their applications*, v. I, Pergamon press, 1964.