Homework 1

Problem 1

Prove that for linear operators A, B, C, D

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB,$$

where $[\ldots]$ and $\{\ldots\}$ are commutator and anticommutator respectively.

Problem 2

Let α and β be two possible quantum states of the same system, and A be a legitimate linear operator. Which of the following expressions are meaningful in the bra-ket formalism?

(i) $\langle\alpha\rangle$ (ii) $\langle\alpha|\beta\rangle^2$ (iii) $|\alpha\rangle\langle\beta|$ (iv) $\langle A|$

(v) $\langle\alpha\rangle A$ (vi) $\alpha|A\rangle$ (vii) $|\alpha\rangle^2$ (iv) A^2

Problem 3

Calculate all possible binary products $\sigma_i\sigma_j$ $(i, j = 1, 2, 3)$ of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Calculate also the commutators and anticommutators of Pauli matrices and their triple product $\sigma_x\sigma_y\sigma_z$.

Problem 4

Prove that for two states α, β of the same system

$$\mathrm{Tr}\ (|\beta\rangle\langle\alpha|) = \langle\alpha|\beta\rangle.$$

Problem 5

Any 2×2 matrix X can be expanded into the "basis" of Pauli matrices

$$X = a_0\sigma_0 + \mathbf{a} \cdot \boldsymbol{\sigma},$$

where $a_{0,1,2,3}$ are complex numbers, σ_0 is a 2×2 unity matrices and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a triple of Pauli matrices ("vector").

Find coefficients $a_{0,1,2,3}$ for a) $X = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$, b) $X = \begin{pmatrix} 2 & 3-i \\ 3+i & 5 \end{pmatrix}$.

c) Prove that $a_{0,1,2,3}$ are real if X is a Hermitian matrix.