# Homework 2

## Problem 1

A skew-Hermitian operator (a.k.a. anti-Hermitian operator) $A$ is an operator satisfying $A^\dagger = -A$.
a) Prove that $A$ can have at most one real eigenvalue (which may be degenerate).
b) Prove that the commutator of two Hermitian operators is skew-Hermitian.

## Problem 2

Prove that the equation $AB - BA = 1$ cannot be satisfied by any finite-dimensional matrices $A$, $B$.

## Problem 3

Let $U$ be a unitary operator. Consider the eigenvalue equation

$$U|\lambda\rangle = \lambda|\lambda\rangle.$$

a) Prove that $\lambda$ is of the form $e^{i\theta}$ with $\theta$ real.
b) Show that if $\lambda \neq \mu$ then $\langle\mu|\lambda\rangle = 0$.

## Problem 4

Consider two operators $A$, $B$ that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\},$$

where

$$A^0\{B\} = B, \qquad A^1\{B\} = [A, B], \qquad A^2\{B\} = [A, [A, B]], \qquad \text{etc.}$$

*Hint:* Consider Taylor expansion in $\lambda$ of the expression $e^{\lambda A} B e^{-\lambda A}$.

## Problem 5

Consider Pauli matrices

$$\sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Find the spectrum and eigenvectors for each of these matrices.
b) Prove that

$$\exp(i\theta\, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}) = \cos\theta + i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\ \sin\theta,$$

where $\hat{\boldsymbol{n}}$ is a unit 3-vector and $\theta$ is some real number.
c) Prove that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors than

$$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b}) + i\boldsymbol{\sigma} \cdot [\boldsymbol{a} \times \boldsymbol{b}].$$

## Problem 6

The operator measuring the spin of a spin-1/2 particle along the axis parallel to a general unit vector $\hat{\boldsymbol{n}}$ is given by

$$S_{\boldsymbol{n}} = \boldsymbol{S} \cdot \hat{\boldsymbol{n}},$$

where $S_i = \sigma_i \hbar/2$ for $i = x, y, z$ and $\sigma_i$ are Pauli matrices.

a) Measurement of an electron's spin along the $z$-axis ($S_z$) using a Stern-Gerlach apparatus gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $\hbar/2$?

b) Measurement of an electron's spin along the axis $\hat{\boldsymbol{n}}$ gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin along the $z$-axis yields $\hbar/2$?

## Problem 7

A spin state is given by $|\alpha\rangle = s_+|+\rangle + s_-|-\rangle$ with some amplitudes $s_\pm$. It is known that in this state $\langle S_z \rangle = 0$ and $\langle S_x \rangle = \hbar/4$.

a) Calculate $\langle S_y \rangle$.

b) What are the possible directions of a quantization axis for this state? That is, what are the possible directions of $\hat{\boldsymbol{n}}$ such that $\langle S_{\boldsymbol{n}} \rangle = \hbar/2$?

## Problem 8 (Sakurai, Ch. 1, Pr. 20)

Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\Big\langle \langle \Delta S_x \rangle^2 \Big\rangle \Big\langle \langle \Delta S_y \rangle^2 \Big\rangle.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for $S_x$ and $S_y$ is not violated.