

Homework 3

Problem 1

Consider the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

- a) Show that A and B commute.
- b) Find the eigenvalues and eigenvectors of A and B .
- c) Find the unitarity transformation which simultaneously diagonalizes A and B .

Problem 2

Consider a Hilbert space \mathcal{H} of two spin-1/2 particles. The following 16 operators form a basis in the vector space of linear operators acting on \mathcal{H} : $\sigma_i \otimes \sigma_j$ with $i, j = 0, 1, 2, 3$ and σ_i are Pauli matrices and a unit matrix (for $i = 0$).

What are the eigenvalues of the operator of the square of the total spin $(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})^2$? Here $\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma} \otimes 1$ and $\boldsymbol{\sigma}^{(2)} = 1 \otimes \boldsymbol{\sigma}$.

A permutation operator is defined as $P(|\alpha\rangle \otimes |\beta\rangle) = |\beta\rangle \otimes |\alpha\rangle$ for any states $|\alpha\rangle$ and $|\beta\rangle$ of the first and the second spin respectively. Express P in terms of basis operators $\sigma_i \otimes \sigma_j$.

Hint: Consider $\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$.

Problem 3

N spin-1/2 particles have a total Hilbert space

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \dots \otimes \mathcal{H}_2^{(N)},$$

where $\mathcal{H}_2^{(i)}$ is the (two-dimensional) Hilbert space of the i th particle.

- a) What is the dimension of \mathcal{H} ?
- b) Define

$$S_z = S_z^{(1)} + S_z^{(2)} + \dots + S_z^{(N)}.$$

What is the spectrum and degeneracy of S_z ?

Problem 4

A particle with spin-1/2 is in an initial state $|+\rangle$ (with spin $+\hbar/2$ along z -axis). Magnetic field B is turned on along positive x -axis at time $t = 0$. What is the state at time $t = \frac{mc}{eB}$?

Problem 5

Consider N -dimensional Hilbert space with orthonormal basis $|n\rangle$, $n = 1, 2, \dots, N$. A Hamiltonian of some quantum system is given by

$$H = -t \sum_{n=1}^N \left(|n+1\rangle\langle n| + |n\rangle\langle n+1| \right),$$

where we use the notation $|N+1\rangle \equiv |1\rangle$ etc.

- Find the spectrum of this Hamiltonian for $N = 3$.
- Find the spectrum of the Hamiltonian for general $N > 2$.
- Can you give an example of the physical system described by this Hamiltonian?

Hint: for b) it is convenient to introduce “Fourier transform” of states $|p\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{ipn} |n\rangle$.

Problem 6

Operators p and q have commutation relation $[p, q] = -i\hbar$. Compute (or simplify)

- $[p^2, q^2]$
- $e^{iap} q e^{-iap}$, where a is some number.
- $e^{ibq} p e^{-ibq}$, where b is some number.
- $\left[\frac{p^2}{2} + V(q), p \right]$ and $\left[\frac{p^2}{2} + V(q), q \right]$, where $V(q)$ is some function.
- $[a, a^\dagger]$, where $a = \frac{1}{\sqrt{2m\hbar}} p - i\sqrt{\frac{m}{2\hbar}} x$, $a^\dagger = \frac{1}{\sqrt{2m\hbar}} p + i\sqrt{\frac{m}{2\hbar}} x$ and m is some number.