Homework 6

Problem 1

Prove the following properties of Dirac's delta-function $\delta(x)$:

a) $\delta(ax) = \frac{1}{|a|}\delta(x)$, where a is some real number.

b) $\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$, where $f(x)$ is some real function of x, prime means derivative with respect to x and summation is over the roots x_i of the equation $f(x) = 0$.

Using b) simplify the following expression $\delta(x^2 - a^2)$.

Problem 2

Consider N-dimensional Hilbert space with orthonormal basis $|n\rangle$, $n = 1, 2, \ldots, N$. A Hamiltonian of some quantum system is given by

$$H = -t \sum_{n=1}^{N} \Big(|n+1\rangle\langle n| + |n\rangle\langle n+1| \Big), \tag{1}$$

where we use the notation $|N+1\rangle \equiv |1\rangle$ etc. Consider an operator

$$T = \sum_{n=1}^{N} |n+1\rangle\langle n|. \tag{2}$$

a) Show that T is a unitary operator and that $T^N = 1$.

b) What is the spectrum of T? Assume that the state $|K\rangle$ is an eigenstate of T: $T\,|K\rangle = e^{iK}\,|K\rangle$ and find allowed values for e^{iK}.

c) Show that T commutes with Hamiltonian so that H and T can be simultaneously diagonalized. What are the eigenenergies in terms of K?

d) Calculate the density of states $\nu(E)/N$ in the limit $N \to \infty$ and plot it as a function of K (You can make a plot "by hand" but it should show the main features).

e) Compare to the results of problem 5 from the homework 3.

Problem 3

Describe (qualitatively) the potentials corresponding to the following scattering matrices (as functions of the wave vector k)

$$S_1(k) = \begin{pmatrix} 0 & e^{ika} \\ e^{-ika} & 0 \end{pmatrix}, \tag{3}$$

$$S_2(k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

$$S_3(k) = \begin{pmatrix} -1 & 0 \\ 0 & -e^{i\eta(k)} \end{pmatrix}. \tag{5}$$

Attention: these are scattering matrices, not transfer matrices.

Problem 4

Consider the following operators

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x), \qquad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x), \tag{6}$$

where p and x are momentum and position operators respectively and m, ω are some constants.

a) Calculate commutator $[a, a^\dagger]$ and anticommutator $\{a, a^\dagger\}$ of these operators using commutation relations of p and x. What are commutators $[a, a]$ and $[a^\dagger, a^\dagger]$?

b) Calculate commutators $[a^\dagger a, a]$ and $[a^\dagger a, a^\dagger]$.

c) Show that if $|n\rangle$ is an eigenstate of $a^\dagger a$, i.e, $a^\dagger a\,|n\rangle = n\,|n\rangle$ with some eigenvalue n than the following states $a^\dagger\,|n\rangle$ and $a\,|n\rangle$ are also eigenstates of $a^\dagger a$. What are corresponding eigenvalues?

d) Calculate commutator $[e^{\alpha a}, a^\dagger]$, where α is some number.

Hint: In d) use the results of problem 4 from the homework 2.