Homework 7

Problem 1
For a 1D harmonic oscillator with mass $m$ and frequency $\omega_0$, calculate:

(i) all matrix elements $\langle n | \hat{x}^3 | n' \rangle$, and

(ii) diagonal matrix elements $\langle n | \hat{x}^4 | n \rangle$,

where $|n\rangle$ are Fock states.

Problem 2
Consider a quantum harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \quad (1)$$

Let us denote $|n\rangle$ with $n = 0, 1, 2, \ldots$ the normalized eigenstates of this Hamiltonian.

a) Find the matrix elements $\langle 5 | x | 6 \rangle$, $\langle 4 | x | 6 \rangle$, $\langle 5 | x^2 | 7 \rangle$, $\langle 11 | p + m\omega x | 11 \rangle$.

b) What is the time evolution of the matrix element $\langle 5 | x | 6 \rangle$?

c) What is the expectation value of the observable $h = p^2 + m^2\Omega^2x^2$ in the eigenstate $|n\rangle$ of the harmonic oscillator (1)?

Problem 3
Consider a particle moving in 1d potential (Morse potential)

$$H = \frac{p^2}{2m} + A\left(e^{-2\alpha x} - 2e^{-\alpha x}\right), \quad (2)$$

where $A$ and $\alpha > 0$ are some constants.

a) Consider the operator $B = -\partial_x + Ce^{-\alpha x} - D$ and its Hermitian conjugate $B^\dagger$. Here $C$ and $D$ are some constants. Find the Hamiltonians $H_- = B^\dagger B$ and $H_+ = BB^\dagger$. Find the values of constants $C$ and $D$ so that $H_-$ is equal to $H$ up to an additive constant.

b) Using the results of a) and solving the equation $B\psi = 0$ find the ground state energy and the ground state wave function for (2).

Problem 4
Using trial wave functions of the form a) $\Psi(x) = Axe^{-\alpha x}$ and b) $\Psi(x) = Bxe^{-\alpha x^2}/2$, where $\alpha$ is a variational parameter find an approximate energy of the ground state of the particle moving in the potential

$$U(x) = \begin{cases} 
    kx, & x > 0 \quad (k > 0), \\
    +\infty, & x < 0.
\end{cases} \quad (3)$$

Compare with the exact result $E_0 \approx 2.338(h^2k^2/2m)^{1/3}$. 
