Homework 9

Reading

LL 26, **EM** Ch 11, **JJS** 3.6.

Problem 1

A Hamiltonian of 2D harmonic oscillator has a form

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2\mathbf{r}^2}{2}, \tag{1}$$

where $\mathbf{r} = (x, y)$ and $\mathbf{p} = -i\hbar\boldsymbol{\nabla} = -i\hbar(\partial_x, \partial_y)$. Find energy levels and eigenfunctions of the oscillator. What is the degeneracy of energy levels?

Hint: Use separation of variables in Cartesian coordinates (x and y).

Problem 2

A Hamiltonian of a plane rotator has a form

$$H = \frac{L^2}{2I}, \tag{2}$$

where L is an operator of an angular momentum $L = -i\hbar\partial_\phi$. For the plane rotator in the state $\Psi = C\cos^2\phi$ find

a) The energy distribution function (probabilities of having different energies) and the expectation value of the energy.

b) The angular momentum distribution function and the expectation value of the angular momentum.

Problem 3

A particle is in an infinitely deep 2D potential well of the form

$$U(r) = \begin{cases} 0, & r \le a, \\ +\infty, & r > a. \end{cases} \tag{3}$$

Find an approximate energy of the ground state of the particle using variational functions

a) $\Psi_0(r) = A(a - r)$;

b) $\Psi_0(r) = B\cos\frac{\pi r}{2a}$.

Compare the obtained results with an exact value $2.88\frac{\hbar^2}{ma^2}$.

Problem 4

The orbital angular momentum of a particle is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{4}$$

where $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$. Starting from the definition of spherical polar coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta \end{aligned} \tag{5}$$

a) Show that the gradient operator can be written in terms of the unit vectors of spherical polar coordinates as

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\,\partial_r + \frac{\hat{\boldsymbol{\phi}}}{r\sin\phi}\partial_\phi + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta. \tag{6}$$

b) Derive the expressions for components of the angular momentum (4) L_x, L_y, L_z in spherical polar coordinates.

c) Derive the expression for $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$ in spherical polar coordinates.

d) Repeat a)-c) for a two-dimensional system in polar coordinates r, ϕ.

Hint: In part a) you can use the expressions for unit vectors

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}. \end{aligned}$$