Homework 1

Problem 1
Prove that for linear operators $A, B, C, D$

a) $[A, BC] = [A, B]C + B[A, C],$

b) $[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB,$

where $[\ldots]$ and $\{\ldots\}$ are commutator and anticommutator respectively.

Problem 2
Let $\alpha$ and $\beta$ be two possible quantum states of the same system, and $A$ be a legitimate linear operator. Which of the following expressions are meaningful in the bra-ket formalism?

(i) $\langle \alpha \rangle$

(ii) $\langle \alpha | \beta \rangle^2$

(iii) $|\alpha\rangle \langle \beta |$

(iv) $\langle A |$

(v) $\langle \alpha | A \rangle$

(vi) $\alpha | A \rangle$

(vii) $|\alpha\rangle^2$

(iv) $A^2$

Problem 3
Calculate all possible binary products $\sigma_i \sigma_j$ $(i, j = 1, 2, 3)$ of Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $

Calculate also the commutators and anticommutators of Pauli matrices and their triple product $\sigma_x \sigma_y \sigma_z.$

Problem 4
Prove that for two states $\alpha, \beta$ of the same system

$\text{Tr} (|\beta\rangle \langle \alpha |) = \langle \alpha | \beta \rangle.$

Problem 5
Any $2 \times 2$ matrix $X$ can be expanded into the “basis” of Pauli matrices

$X = a_0 \sigma_0 + a \cdot \sigma,$

where $a_{0,1,2,3}$ are complex numbers, $\sigma_0$ is a $2 \times 2$ unity matrices and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a triple of Pauli matrices (“vector”).

Find coefficients $a_{0,1,2,3}$ for a) $X = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix},$ b) $X = \begin{pmatrix} 2 & 3 - 2i \\ 3 + 2i & 5 \end{pmatrix}.$

c) Prove that $a_{0,1,2,3}$ are real if $X$ is a Hermitian matrix.