Homework 2

Problem 1
A skew-Hermitian operator (a.k.a. anti-Hermitian operator) $A$ is an operator satisfying $A^\dagger = -A$.

a) Prove that $A$ can have at most one real eigenvalue (which may be degenerate).

b) Prove that the commutator of two Hermitian operators is skew-Hermitian.

Problem 2
Prove that the equation $AB - BA = 1$ cannot be satisfied by any finite-dimensional matrices $A, B$.

Problem 3
Let $U$ be a unitary operator. Consider the eigenvalue equation

$$ U|\lambda\rangle = \lambda|\lambda\rangle. $$

a) Prove that $\lambda$ is of the form $e^{i\theta}$ with $\theta$ real.

b) Show that if $\lambda \neq \mu$ then $\langle \mu|\lambda \rangle = 0$.

Problem 4
Consider two operators $A, B$ that do not necessarily commute. Show that

$$ e^ABe^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}, $$

where

$$ A^0\{B\} = B, \quad A^1\{B\} = [A, B], \quad A^2\{B\} = [A, [A, B]], \quad \text{etc.} $$

*Hint:* Consider Taylor expansion in $\lambda$ of the expression $e^{\lambda A}B e^{-\lambda A}$.

Problem 5
Consider Pauli matrices

$$ \sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

a) Find the spectrum and eigenvectors for each of these matrices.

b) Prove that if $a$ and $b$ are two vectors than

$$ (\sigma \cdot a)(\sigma \cdot b) = (a \cdot b) + i \sigma \cdot [a \times b]. $$

c) Prove that

$$ \exp(i\theta \sigma \cdot \hat{n}) = \cos \theta + i \sigma \cdot \hat{n} \sin \theta, $$

where $\hat{n}$ is a unit 3-vector and $\theta$ is some real number.
Problem 6

The operator measuring the spin of a spin-1/2 particle along the axis parallel to a general unit vector $\hat{n}$ is given by

$$S_n = S \cdot \hat{n},$$

where $S_i = \sigma_i \hbar / 2$ for $i = x, y, z$ and $\sigma_i$ are Pauli matrices.

a) Measurement of an electron’s spin along the $z$-axis ($S_z$) using a Stern-Gerlach apparatus gives the eigenvalue $\hbar / 2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ yields $\hbar / 2$?

b) Measurement of an electron’s spin along the axis $\hat{n}$ gives the eigenvalue $\hbar / 2$. What is the probability that a subsequent measurement of the spin along the $z$-axis yields $\hbar / 2$?

Problem 7

A spin state is given by $|\alpha\rangle = s_+ |+\rangle + s_- |-\rangle$ with some amplitudes $s_{\pm}$. It is known that in this state $\langle S_z \rangle = 0$ and $\langle S_x \rangle = \hbar / 4$.

a) Calculate $\langle S_y \rangle$.

b) What are the possible directions of a quantization axis for this state? That is, what are the possible directions of $\hat{n}$ such that $\langle S_n \rangle = \hbar / 2$?