Homework 4

Problem 1
A Hamiltonian of some quantum system is given by
\[ H = e^{i\Phi}|2\rangle\langle 1| + e^{i\Phi}|3\rangle\langle 2| + e^{i\Phi}|1\rangle\langle 3| + h.c., \]
where the states \(|1\rangle, |2\rangle\) and \(|3\rangle\) for an orthonormal basis in the Hilbert space of the system, \(\Phi\) is some real number and “h.c.” means Hermitian conjugate.

a) Find the spectrum of this Hamiltonian.
b) Find the eigenstates of this Hamiltonian.
c) Find a time-dependent solution \(|\Psi(t)\rangle\) of the Schrödinger equation with this Hamiltonian corresponding to an initial state \(|\Psi(0)\rangle = |1\rangle\).

Problem 2
Prove the following properties of Dirac’s delta-function \(\delta(x)\):

a) \(\delta(ax) = \frac{1}{|a|}\delta(x)\), where \(a\) is some real number.
b) \(\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x-x_i)\), where \(f(x)\) is some real function of \(x\), prime means derivative with respect to \(x\) and summation is over the roots \(x_i\) of the equation \(f(x) = 0\).
c) Using b) simplify the following expression \(\delta(x^2 - a^2)\).

Problem 3
The derivative of an operator \(A(\xi)\) depending explicitly on a continuous parameter \(\xi\) is by definition
\[ \frac{dA}{d\xi} = \lim_{\epsilon\to 0} \frac{A(\xi + \epsilon) - A(\xi)}{\epsilon}. \]

a) Using this definition show that if two operators \(A\) and \(B\) are differentiable, then
\[ \frac{d}{d\xi}(AB) = \frac{dA}{d\xi}B + A\frac{dB}{d\xi}. \]
In particular
\[ \frac{d}{d\xi}(A^2) = \frac{dA}{d\xi}A + A\frac{dA}{d\xi}. \]
b) Show that if \(A\) is differentiable and possesses an inverse, one has
\[ \frac{d}{d\xi}A^{-1} = -A^{-1}\frac{dA}{d\xi}A^{-1}. \]

Problem 4
Show that if the operator \(U(t)\), differentiable with respect to a parameter \(t\), is unitary, the operator
\[ H(t) \equiv i\frac{d}{dt}U^\dagger \]
is necessarily Hermitian.