Homework 4

Problem 1

A Hamiltonian of some quantum system is given by

$$H = e^{i\Phi}|2\rangle\langle 1| + e^{i\Phi}|3\rangle\langle 2| + e^{i\Phi}|1\rangle\langle 3| + h.c.,$$

where the states $|1\rangle$, $|2\rangle$ and $|3\rangle$ for an orthonormal basis in the Hilbert space of the system, Φ is some real number and "$h.c.$" means Hermitian conjugate.

a) Find the spectrum of this Hamiltonian.

b) Find the eigenstates of this Hamiltonian.

c) Find a time-dependent solution $|\Psi(t)\rangle$ of the Schrodinger equation with this Hamiltonian corresponding to an initial state $|\Psi(0)\rangle = |1\rangle$.

Problem 2

Prove the following properties of Dirac's delta-function $\delta(x)$:

a) $\delta(ax) = \frac{1}{|a|}\delta(x)$, where a is some real number.

b) $\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$, where $f(x)$ is some real function of x, prime means derivative with respect to x and summation is over the roots x_i of the equation $f(x) = 0$.

c) Using b) simplify the following expression $\delta(x^2 - a^2)$.

Problem 3

The derivative of an operator $A(\xi)$ depending explicitly on a continuous parameter ξ is by definition

$$\frac{dA}{d\xi} = \lim_{\epsilon \to 0} \frac{A(\xi + \epsilon) - A(\xi)}{\epsilon}.$$

a) Using this definition show that if two operators A and B are differentiable, then

$$\frac{d}{d\xi}(AB) = \frac{dA}{d\xi}B + A\frac{dB}{d\xi}.$$

In particular

$$\frac{d}{d\xi}(A^2) = \frac{dA}{d\xi}A + A\frac{dA}{d\xi}.$$

b) Show that if A is differentiable and possesses and inverse, one has

$$\frac{d}{d\xi}A^{-1} = -A^{-1}\frac{dA}{d\xi}A^{-1}.$$

Problem 4

Show that if the operator $U(t)$, differentiable with respect to a parameter t, is unitary, the operator

$$H(t) \equiv i\frac{dU}{dt}U^\dagger$$

is necessarily Hermitean.