Homework 5

Problem 1

a) Find the potential profile $U(x)$ for which the following wavefunction,

$$\Psi(x,t) = Ce^{-ax^2 - ibt}.$$

(with real constants a and b, $a > 0$), satisfies the Schrödinger equation for a particle with mass m.
b) Normalize the wavefunction to $P = 1$, find $\langle x \rangle$, $\langle p_x \rangle$, δx, and δp_x, and compare the product $\delta x \, \delta p_x$ with Heisenberg's uncertainty relation.

Problem 2

A quantum particle of the mass m moves in an infinite potential well $U(x) = 0$ if $-a < x < a$ and $U(x) = \infty$ if $|x| > a$.

a) Write down the spectrum E_n and normalized eigenfunctions $\psi_n(x)$, $n = 0, 1, 2, \ldots$ for this system.

b) What is the force exerted by a particle on the walls of the potential well in the ground state?

c) Suppose that the particle is initially in the (unnormalized) state $\psi(x, t = 0) = x$. What is the probability to find the particle in the ground state ($n = 0$) in time t? What is the probability to find the particle in the first excited state $n = 1$ in time t? What is the probability to find this particle in the state $n = 24$ at the same time?

Hint: in part b) consider the dependence of the ground state energy on the size of the well.

Problem 3

A quantum particle of the mass m moves in a potential well $U(x) = -U_0$ if $-a < x < a$ and $U(x) = 0$ if $|x| > a$. Here $U_0 > 0$ is some constant.

a) Write down the equations determining the spectrum and eigenstates of bound states in this problem. (the equations are transcendental and can be solved only numerically in this case).

b) In the limit of a very shallow well $U_0 \ll \frac{\hbar^2}{ma^2}$ find an approximate energy of the ground state. How many bound states do exist in this case?

c) Write down an approximate wave function of the ground state in the case of a very shallow well.

Problem 4

Consider the following operators

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - im\omega x), \qquad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x), \tag{1}$$

where p and x are momentum and position operators respectively and m, ω are some constants.

a) Calculate commutator $[a, a^\dagger]$ and anticommutator $\{a, a^\dagger\}$ of these operators using commutation relations of p and x. What are commutators $[a, a]$ and $[a^\dagger, a^\dagger]$?

b) Calculate commutators $[a^\dagger a, a]$ and $[a^\dagger a, a^\dagger]$.

c) Show that if $|n\rangle$ is an eigenstate of $a^\dagger a$, i.e, $a^\dagger a\,|n\rangle = n\,|n\rangle$ with some eigenvalue n than the following states $a^\dagger\,|n\rangle$ and $a\,|n\rangle$ are also eigenstates of $a^\dagger a$. What are corresponding eigenvalues?

d) Calculate commutator $[e^{\alpha a}, a^\dagger]$, where α is some number.

Hint: In d) use the results of problem 2.4.