Homework 6

Problem 1

a) Find the potential profile $U(x)$ for which the following wavefunction,

$$\Psi(x, t) = Ce^{-ax - ibt}$$

(with real and constant $a > 0$ and $b$), satisfies the Schrödinger equation for a particle with mass $m$.

b) Normalize the wavefunction to $P = 1$, find $\langle x \rangle$, $\langle p_x \rangle$, $\delta x$, and $\delta p_x$, and compare the product $\delta x \delta p_x$ with Heisenberg’s uncertainty relation.

Problem 2

Consider $N$-dimensional Hilbert space with orthonormal basis $|n\rangle$, $n = 1, 2, \ldots, N$. A Hamiltonian of some quantum system is given by

$$H = -t \sum_{n=1}^{N} \left( |n+1\rangle\langle n| + |n\rangle\langle n+1| \right), \tag{1}$$

where we use the notation $|N+1\rangle \equiv |1\rangle$ etc. Consider an operator

$$T = \sum_{n=1}^{N} |n+1\rangle\langle n|. \tag{2}$$

a) Show that $T$ is a unitary operator and that $T^N = 1$.

b) What is the spectrum of $T$? Assume that the state $|K\rangle$ is an eigenstate of $T$: $T |K\rangle = e^{iK} |K\rangle$ and find allowed values for $e^{iK}$.

c) Show that $T$ commutes with Hamiltonian so that $H$ and $T$ can be simultaneously diagonalized. What are the eigenenergies in terms of $K$?

d) Calculate the density of states $\nu(E)/N$ in the limit $N \to \infty$ and plot it as a function of $K$ (You can make a plot “by hand” but it should show the main features).

Problem 3

Find the transmission coefficient $T(E)$ as a function of the energy $E > 0$ of the particle scattered by a potential

$$U(x) = -W\delta(x).$$

Problem 4

Find the number of bound states as a function of the parameter $\xi = mWa/\hbar^2$ for the particle moving in a potential $U(x)$ of the form

$$U(x) = \begin{cases} \infty, & x < 0, \\ -W\delta(x-a), & x > 0. \end{cases}$$
Problem 5
Find the values of energies at which particles are not reflected by the potential barrier

\[ U(x) = W [\delta(x) + \delta(x - a)]. \]

Problem 6
Assume that \( p_{x,y,z} \) and \( x, y, z \) are Hermitian operators having commutators \( [p_x, x] = [p_y, y] = [p_z, z] = -i \) and other commutators are trivial (zero). Calculate

a) \( [p_x, x^2 y + xy^2], \)

b) \( (xp_x + yp_y)^\dagger, \)

c) \( \left( \frac{x}{r} p_x + \frac{y}{r} p_y \right)^\dagger, \) where \( r = \sqrt{x^2 + y^2}, \)

d) \( [xp_y - yp_x, f(r)], \) where \( r = \sqrt{x^2 + y^2} \) and \( f(r) \) is some function,

e) \( [xp_y - yp_x, xp_z - zp_x], \)

f) \( \exp \left\{ \alpha \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix} \right\}, \) where \( \alpha \) is some complex number.

Hint: in f) you can either diagonalize the matrix first or express the exponent in terms of Pauli matrices and consider the Taylor expansion of the exponential function.