Homework 6

Problem 1

a) Find the potential profile $U(x)$ for which the following wavefunction,

$$\Psi(x,t) = Ce^{-a|x|-ibt}$$

(with real and constant $a > 0$ and b), satisfies the Schrödinger equation for a particle with mass m.
b) Normalize the wavefunction to $P = 1$, find $\langle x \rangle$, $\langle p_x \rangle$, δx, and δp_x, and compare the product $\delta x\, \delta p_x$ with Heisenberg's uncertainty relation.

Problem 2

Consider N-dimensional Hilbert space with orthonormal basis $|n\rangle$, $n = 1, 2, \ldots, N$. A Hamiltonian of some quantum system is given by

$$H = -t \sum_{n=1}^{N} \Big(|n+1\rangle\langle n| + |n\rangle\langle n+1| \Big), \tag{1}$$

where we use the notation $|N+1\rangle \equiv |1\rangle$ etc. Consider an operator

$$T = \sum_{n=1}^{N} |n+1\rangle\langle n|. \tag{2}$$

a) Show that T is a unitary operator and that $T^N = 1$.
b) What is the spectrum of T? Assume that the state $|K\rangle$ is an eigenstate of T: $T\,|K\rangle = e^{iK}\,|K\rangle$ and find allowed values for e^{iK}.
c) Show that T commutes with Hamiltonian so that H and T can be simultaneously diagonalized. What are the eigenenergies in terms of K?
d) Calculate the density of states $\nu(E)/N$ in the limit $N \to \infty$ and plot it as a function of K (You can make a plot "by hand" but it should show the main features).

Problem 3

Find the transmission coefficient $T(E)$ as a function of the energy $E > 0$ of the particle scattered by a potential

$$U(x) = -W\delta(x).$$

Problem 4

Find the number of bound states as a function of the parameter $\xi = mWa/\hbar^2$ for the particle moving in a potential $U(x)$ of the form

$$U(x) = \begin{cases} \infty, & x < 0, \\ -W\delta(x-a), & x > 0. \end{cases}$$

Problem 5

Find the values of energies at which particles are not reflected by the potential barrier

$$U(x) = W\left[\delta(x) + \delta(x-a)\right].$$

Problem 6

Assume that $p_{x,y,z}$ and x, y, z are Hermitian operators having commutators $[p_x, x] = [p_y, y] = [p_z, z] = -i$ and other commutators are trivial (zero). Calculate

a) $[p_x, x^2y + xy^2]$,

b) $\left(xp_x + yp_y\right)^\dagger$,

c) $\left(\frac{x}{r}p_x + \frac{y}{r}p_y\right)^\dagger$, where $r = \sqrt{x^2+y^2}$,

d) $[xp_y - yp_x, f(r)]$, where $r = \sqrt{x^2+y^2}$ and $f(r)$ is some function,

e) $[xp_y - yp_x, xp_z - zp_x]$,

f) $\exp\left\{\alpha \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix}\right\}$, where α is some complex number.

Hint: in f) you can either diagonalize the matrix first or express the exponent in terms of Pauli matrices and consider the Taylor expansion of the exponential function.