Physics 501: Classical Mechanics

Read: LL 40-45;

Homework 8

Exercise 1
A Hamiltonian of a charged particle moving in a constant uniform magnetic field \( \vec{B} \) is given by

\[
H = \frac{(\vec{p} - \frac{e}{c} \vec{A})^2}{2m},
\]
where \( \vec{A} (\vec{r}) \) is a vector potential corresponding to the constant magnetic field \( (\vec{\nabla} \times \vec{A}) = \vec{B} \).

a) Evaluate the Poisson brackets \([v_i, v_j]\) of the Cartesian components of the velocity of the particle \( v_i = \dot{x}_i, i = 1, 2, 3 \).

b) Rewrite the Hamiltonian in terms of particle’s velocity \( \vec{v} \) and find \( \frac{d\vec{v}}{dt} \) commuting the Hamiltonian with \( \vec{v} \) and using the results of a).

Exercise 2 (G 9.30)
A system of two degrees of freedom is described by the Hamiltonian

\[
H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2.
\]
Show that \( F_1 = \frac{p_1 - a q_1}{q_2} \) and \( F_2 = q_1 q_2 \) are constants of motion. Are there any other independent algebraic constants of motion? Can any be constructed from Jacobi’s identity?

Exercise 3
Hamiltonian of the Kepler problem is given by

\[
H = \frac{\vec{p}^2}{2m} + \frac{\alpha}{r}.
\]

a) Show from the Poisson bracket condition for conserved quantities that the Laplace-Runge-Lenz vector \( \vec{A} = \vec{p} \times \vec{M} + \alpha m \hat{r} \) is a constant. Here \( \vec{M} = \vec{r} \times \vec{p} \) is an angular momentum and \( \hat{r} = \vec{r}/r \).

b) Derive \([A_i, A_j]\) - the Poisson’s bracket of Cartesian components of \( \vec{A} \).

Hint: See G. Section 9-7.
Exercise 4 (G 9.4)
Show directly that the transformation \( Q = \log(\frac{1}{q} \sin p), \ P = q \cot p \) is canonical.

*Exercise 5 (G 9.8)
Prove directly that the transformation

\[
\begin{align*}
Q_1 &= q_1, & P_1 &= p_1 - 2p_2, \\
Q_2 &= p_2, & P_2 &= -2q_1 - q_2
\end{align*}
\]

is canonical and find a generating function.