

## Homework 1.

### Exercise 1: Bohr-van Leeuwen theorem

Prove the Bohr-van Leeuwen theorem averaging directly the classical magnetization

$$\mathbf{M} = \frac{1}{Z} \int \prod_{k=1}^N d\mathbf{p}_k d\mathbf{r}_k \hat{\mathbf{M}}(\mathbf{r}_k, \dot{\mathbf{r}}_k) e^{-\beta\mathcal{H}}, \quad (1)$$

with magnetization

$$\hat{\mathbf{M}} = \sum_{k=1}^N \frac{e_k}{2c} [\mathbf{r}_k \times \dot{\mathbf{r}}_k], \quad (2)$$

and classical Hamiltonian

$$\mathcal{H} = \sum_{k=1}^N \frac{1}{2m_k} \left( \vec{p}_k - \frac{e_k}{c} \vec{A}_k \right)^2 + V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N). \quad (3)$$

*Hint.* Use Hamilton equations of motion  $\dot{p} = -\frac{\partial\mathcal{H}}{\partial q}$ ,  $\dot{q} = \frac{\partial\mathcal{H}}{\partial p}$ .

### Exercise 2: Spin operators

Let us introduce Pauli matrices  $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$  with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4)$$

and electron creation and annihilation operators  $c_\alpha, c_\alpha^\dagger$  with  $\alpha = 1, 2$  corresponding to spin up and down. (Anti)commutation relations are

$$\{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0, \quad (5)$$

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}. \quad (6)$$

a) Using these commutation relations show that spin operators  $\hat{S}^a = \frac{1}{2} c_\alpha^\dagger \sigma_{\alpha\beta}^a c_\beta$  satisfy spin commutation relations  $[\hat{S}^a, \hat{S}^b] = i\epsilon^{abc} \hat{S}^c$

b) Show that the full spin squared  $\hat{S}^2 = \hat{S}^a \hat{S}^a = \frac{3}{4}$  corresponding to  $S = \frac{1}{2}$ .

### Exercise 3: Two-electron states

Using fermion anticommutation relations show that the following two-electron states

$$\begin{aligned} |1, 1\rangle &= c_{1\uparrow}^\dagger c_{2\uparrow}^\dagger |0\rangle, \\ |1, 0\rangle &= \frac{1}{\sqrt{2}}(c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger + c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger) |0\rangle, \\ |1, -1\rangle &= c_{1\downarrow}^\dagger c_{2\downarrow}^\dagger |0\rangle, \\ |0, 0\rangle &= \frac{1}{\sqrt{2}}(c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger) |0\rangle \end{aligned}$$

are the eigenstates of an operator  $\hat{S}^2$  with eigenvalues  $S(S+1)$  and of operator  $\hat{S}^z$  by directly acting by these operators on the states. Find corresponding eigenvalues  $S$  and  $S^z$  for each state. Total spin operators are defined as

$$\hat{S}^a = \sum_{k=1,2; \alpha, \beta=1,2} \frac{1}{2} c_{k,\alpha}^\dagger \sigma_{\alpha\beta}^a c_{k,\beta}, \quad (7)$$

where  $c_{k,\alpha}^\dagger$  is a creation operator which creates an electron in the one-electron state  $k = 1, 2$  with spin  $\alpha$ . Vacuum state  $|0\rangle$  is defined so that  $c_{k,\alpha}|0\rangle = 0$ .

### Exercise 4: Double exchange model on two lattice sites

Consider the double exchange model on two lattice sites ( $k = 1, 2$ )

$$\mathcal{H} = t(c_{1,\alpha}^\dagger c_{2,\alpha} + c_{2,\alpha}^\dagger c_{1,\alpha}) - J_H \sum_{k=1,2} \vec{S}_k \cdot \frac{1}{2} c_{k,\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{k,\beta}, \quad (8)$$

where summation over spin indices is assumed.  $\vec{S}_{1,2}$  are classical ( $S \gg 1$ ) core spins at lattice sites 1, 2 respectively. Conduction electrons can hop from one lattice site to another (first term in the Hamiltonian) and interact with core spins by ferromagnetic exchange (Hund's rule) interaction (second term). Assume that there is just one electron on those two sites  $\sum_{k=1,2} c_{k,\alpha}^\dagger c_{k\alpha} = 1$ . In the limit  $J_H \rightarrow \infty$  find the lowest eigenvalue  $E_{ground}(\vec{S}_1, \vec{S}_2)$  of the electron living at those sites if the core spins do not change in time. Using adiabatic approximation (which is justified by small parameter  $1/S$ ) write down an effective Hamiltonian for just core spins ("induced by electron") as

$$\mathcal{H}_{ad} = E_{ground}(\vec{S}_1, \vec{S}_2). \quad (9)$$

Is this interaction ferro- or antiferromagnetic? When can this interaction be replaced by Heisenberg model?

*Hint 1.* It is convenient to direct, say,  $\vec{S}_1$  along  $z$ -axis.

*Hint 2.* Use canonic transformation  $c_{2,\alpha} \rightarrow U_{\alpha\beta} c_{2,\beta}$  such that  $U^{-1} \vec{S}_2 \vec{\sigma} U = S \sigma^z$ .