Homework 3.

Exercise 1: Mean field free energy from microscopics

Consider a three-dimensional classical easy-axis ferromagnet described by microscopic Hamiltonian:

$$\mathcal{H} = -\sum_{i,j} J_{ij}\vec{S}_i\vec{S}_j - D\sum_i (S_i^z)^2 - \sum_i \vec{h}\vec{S}_i, \quad (1)$$

where $\vec{S}_i^2 = S^2$ is a classical spin at site i of a simple cubic lattice, exchange constants $J_{ij} = J$ are non zero for nearest neighbors only, and anisotropy is small $J \gg D > 0$.

a) Using Weiss molecular field approach (mean field) find the critical temperature of magnetic ordering phase transition and equations determining spontaneous magnetization at $T < T_c$.

b) Show that this spontaneous magnetization can be obtained at T close to T_c by minimizing the following free energy

$$f = \frac{1}{2\chi_{zz}}m_z^2 + \frac{1}{2\chi_\perp}(m_x^2 + m_y^2) + bm^4, \quad (2)$$

where $m^2 = m_z^2 + m_x^2 + m_y^2$. Express $\chi_{zz}, \chi_\perp, b$ in terms of T_c and D.

Exercise 2: Gradient terms

In RPA approximation calculate the gradient part of free energy for the same model as in the previous exercise. Express the constant g in front of gradient term of free energy in terms of microscopic parameters J, D or T_c.

Hint. Obtain high temperature susceptibility in RPA, write gradient part as $\frac{1}{2}m_a(-q)[\hat{\chi}(q)]^{-1}_{ab}m_b(q)$, and expand $\chi(q)$ at small q.

Exercise 3: Isotropic ferromagnet

Neglect anisotropy D in previous exercises. Write down the condition of applicability of the mean field in the vicinity of critical temperature (Ginzburg-Levanyuk criterion). Is there any small parameter which guarantees the existence of the range where mean field theory works well? Will fluctuation region expand or shrink if one increases (decreases) the dimensionality of the order parameter (think of spin as of a vector in N-dimensional space)?

Exercise 4: Ising ferromagnet

If ferromagnet has a weak easy-axis anisotropy $D \ll J$ one can use Ising model close enough to the phase transition point. How close should temperature be to T_c to use Ising model instead of full anisotropic free energy? What happens with Ginzburg-Levanyuk criterion in this limit?

Exercise 5: Domain wall

Consider the same magnetic system as in exercise 1 at temperature very close but slightly lower than T_c. One can consider *domain wall* – two-dimensional surface separating half-spaces with opposite uniform magnetizations.

a) Minimizing mean field free energy obtained in exercises 1,2 find the "shape" of a domain wall. Namely, find the configuration $m_z(z)$ where z is the axis perpendicular to domain wall and m_z is a component of magnetization along the easy axis. Boundary conditions defining domain wall are, obviously, $m_z(\pm\infty) = \mp m_0$, where m_0 is an equilibrium uniform magnetization. What is the typical width of the domain wall? What is the free energy of domain wall per unit square?

b) What happens if temperature is further away from T_c, namely, in the range where all three components of magnetization play role. To understand it better, find the shape of domain wall at zero temperature assuming that $D \ll J$. What is the typical width of the domain wall? What is the energy of domain wall per unit square?

Hint. Prove that in the continuum approximation (1) becomes

$$\mathcal{H} = \int \frac{d^3x}{a^3} \left[Ja^2(\vec{\nabla} S^a)^2 - D(S^z)^2 \right], \tag{3}$$

parameterize spin as $\vec{S} = S(\sin\theta(z), 0, \cos\theta(z))$ and minimize the energy with respect to $\theta(z)$.

Exercise 6: Spin vortex

Assume now that we have an *easy-plane* ferromagnet described by (1) but with $J \gg -D > 0$. Then spins prefer to lie in the xy plane and there exist another topological defect – spin vortex instead of domain wall. Estimate (very roughly, by analogy with domain wall, no calculations) the size of the core of the vortex. How do you define this core? What happens with magnetization (qualitatively) inside the core? Consider two separate cases: temperature is in the close vicinity of the critical temperature and temperature is zero.