2/5/2004. **Due:** Thursday, February 19, 2004 (in class).

Physics 682: Quantum Magnetism

Problems with stars are not for credit and will NOT be graded.

Homework 1

Exercise 1: Quantum Langevin paramagnetism

We describe an atom with nonzero magnetic moment in magnetic field by the Hamiltonian

$$H = -\mu_{\mathrm{B}} g J_z B,$$

where g is Lande factor and $J_z = -J, \ldots J-1, J$ is a projection of total angular momentum of the atom $\hat{J}$ on the direction of magnetic field. Write down the partition function for such an atom (system of non-interacting atoms) at temperature T. Find an average magnetic moment of an atom from this partition function. Present the result in the form $M = \mu B_J(x)$, where $x = \mu_B g J B / T$ is a dimensionless strength of magnetic field. Write down an explicit expression for a Brillouin function $B_J(x)$. What is $B_{1/2}(x)$, $B_\infty(x)$? Find the magnetic susceptibility per atom and compare the result with the Langevin's result for a classical magnetic moment.

Exercise 2: Weiss ferromagnetism

Using classical Weiss equation for spontaneous magnetic moment in ferromagnets

$$M = \mu L \left(\frac{\mu \lambda M}{T} \right)$$

find the value of Curie temperature T_c below which magnetic order exists. Find an analytic dependence of magnetization $M(T)$ on temperature T for temperatures very close to the Curie temperature. In the final result replace an unknown constant λ by T_c. Sketch the dependence M vs. T for all temperatures.

Hint: Expand $L(x)$ up to the second nonvanishing term.

Exercise 3: Weiss antiferromagnetism

Assume that the magnetic moment μ of an atom in a crystal "feels" an internal magnetic field $B_{int} = -\lambda M$, where M its average magnetic moment and $\lambda > 0$ is some constant. a) Derive Curie-Weiss law for antiferromagnets by calucating $M(B = 0, T)$ using Langevin formulas at high temperatures. b) What happens if the temperature is lower than the critical temperature T_c? What is the value of

T_c in terms of λ. c) Plot the dependence of $M(B = 0, T)$. d) Find the magnetic susceptibility χ at $T = 0$ for this system. Is it para- or diamagnetic?

Hint: In d) take into account that local magnetic moments will arrange in the direction approximately orthogonal to the direction of an applied uniform magnetic field B.

Exercise 4: Spin operators

Let us introduce Pauli matrices $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

and electron creation and annihilation operators $c_\alpha, c_\alpha^\dagger$ with $\alpha = 1, 2$ corresponding to spin up and down. (Anti)commutation relations are

$$\{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0, \tag{2}$$
$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}. \tag{3}$$

a) Using these commutation relations show that spin operators $\hat{S}^a = \frac{1}{2} c_\alpha^\dagger \sigma_{\alpha\beta}^a c_\beta$ satisfy spin commutation relations $[\hat{S}^a, \hat{S}^b] = i\epsilon^{abc} \hat{S}^c$

b) Show that the full spin squared $(\vec{\hat{S}})^2 = \hat{S}^a \hat{S}^a = \frac{3}{4}$ corresponds to $S = \frac{1}{2}$.

Exercise 5: Two-electron states

Using fermion anticommutation relations show that the following two-electron states

$$|1, 1\rangle = c_{1\uparrow}^\dagger c_{2\uparrow}^\dagger |0\rangle,$$
$$|1, 0\rangle = \frac{1}{\sqrt{2}} (c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger + c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger)|0\rangle,$$
$$|1, -1\rangle = c_{1\downarrow}^\dagger c_{2\downarrow}^\dagger |0\rangle,$$
$$|0, 0\rangle = \frac{1}{\sqrt{2}} (c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger - c_{1\downarrow}^\dagger c_{2\uparrow}^\dagger)|0\rangle$$

are the eigenstates of an operator $\vec{\hat{S}}^2$ with eigenvalues $S(S + 1)$ and of operator $\hat{S}^z$ by directly acting by these operators on the states. Find corresponding eigenvalues S and S^z for each state. Total spin operators are defined as

$$\hat{S}^a = \sum_{k=1,2;\ \alpha,\beta=1,2} \frac{1}{2} c_{k,\alpha}^\dagger \sigma_{\alpha\beta}^a c_{k,\beta}, \tag{4}$$

where $c_{k,\alpha}^\dagger$ is a creation operator which creates an electron in the one-electron state $k = 1, 2$ with spin α. Vacuum state $|0\rangle$ is defined so that $c_{k,\alpha}|0\rangle = 0$.

*Exercise 6: Double exchange model on two lattice sites

Consider the double exchange model on two lattice sites ($k = 1, 2$)

$$\mathcal{H} = t(c^\dagger_{1,\alpha}c_{2,\alpha} + c^\dagger_{2,\alpha}c_{1,\alpha}) - J_H \sum_{k=1,2} \vec{S}_k \cdot \frac{1}{2}c^\dagger_{k,\alpha}\vec{\sigma}_{\alpha\beta}c_{k,\beta}, \qquad (5)$$

where summation over spin indices is assumed. $\vec{S}_{1,2}$ are classical ($S \gg 1$) core spins at lattice sites $1, 2$ respectively. Conduction electrons can hop from one lattice site to another (first term in the Hamiltonian) and interact with core spins by ferromagnetic exchange (Hund's rule) interaction (second term). Assume that there is just one electron on those two sites $\sum_{k=1,2} c^\dagger_{k,\alpha}c_{k\alpha} = 1$. In the limit $J_H \to \infty$ find the lowest eigenvalue $E_{ground}(\vec{S}_1, \vec{S}_2)$ of the electron living at those sites if the core spins do not change in time. Using adiabatic approximation (which is justified by small parameter $1/S$) write down an effective Hamiltonian for just core spins ("induced by electron") as

$$\mathcal{H}_{ad} = E_{ground}(\vec{S}_1, \vec{S}_2). \qquad (6)$$

Is this interaction ferro- or antiferromagnetic? When can this interaction be replaced by Heisenberg interaction?

Hint 1. It is convenient to direct, say, $\vec{S}_1$ along z-axis.

Hint 2. Use canonic transformation $c_{2,\alpha} \to U_{\alpha\beta}c_{2,\beta}$ such that $U^{-1}\vec{S}_2\vec{\sigma}U = S\sigma^z$.

For original papers on double exchange see

1. C. Zener, Phys. Rev. **82**, 403 (1951).

2. P.W. Anderson and H. Hasegawa, Phys. Rev. **100**, 675 (1955).

3. P.G. de Gennes, Phys. Rev. **118**, 141 (1960).