Physics 682: Quantum Magnetism

Problems with stars are not for credit and will NOT be graded.

Homework 2

Exercise 1: High temperature susceptibility

Consider the classical Hamiltonian:

$$\mathcal{H} = \sum_{i,j} J_{ij} \vec{S}_i \vec{S}_j + D \sum_i (S_i^z)^2, \tag{1}$$

where i, j are sites of three-dimensional stacked triangular lattice (simple Bravais lattice with lattice vectors $e_1 = (1, 0, 0)$, $e_2 = (1/2, \sqrt{3}/2, 0)$, $e_3 = (0, 0, 1)$. Non-zero exchange integrals are given by $J_{i,i\pm e_3} = J > 0$, $J_{i,i\pm e_{1,2}} = J_{i,i\pm(e_1-e_2)} = J' > 0$, and $D > 0$ is an easy plane anisotropy constant. Consider the following ratios between constants $J : D : J' = 500 : 5 : 1$.

a) In RPA approximation calculate the high temperature static susceptibility at wave vector q. Consider both cases: magnetic field is in (easy) plane and perpendicular to the plane.

b) What is the wave vector Q at which instability occurs when lowering the temperature? What is the critical temperature at which this transition occurs?

*c) Do you think the actual critical temperature is close to the one obtained in RPA? What is better estimate for critical temperature?

Hint. Remember that in purely one dimensional system ordering does not occur at all.

d) Find the uniform susceptibility at high temperatures. What happens with this susceptibility at the critical temperature found in b)?

Exercise 2: Classical ground state

For the spin Hamiltonian given in exercise 1 find the classical ground state (spin configuration minimizing the Hamiltonian). Find the uniform susceptibility $\chi_\parallel$ (magnetic field is applied parallel to z-axis) at zero temperature.

Hint. Use the value Q obtained in exercise 1.

*Exercise 3: In-plane magnetic field

Find $\chi_\perp$ (magnetic field is in plane) at zero temperature for the system from exercises 1,2. Find $\chi_\perp$ (magnetic field is in plane) at zero temperature and at finite magnetic field $J'S \ll h \ll JS$.

*Exercise 4: Spin waves

For the same system find the spectrum of spin waves at very low temperatures (assuming that spins are classical). How many branches of spin waves have you obtained? How many Goldstone modes have you found? Find (or estimate) the thermal corrections to the average spin on lattice site (calculate $\langle \vec{S}_i \rangle$ at very low temperature. Which spin wave modes contribute the most to these thermal corrections?

Exercise 5: Mean field free energy from microscopics

Consider a three-dimensional classical easy-axis ferromagnet described by microscopic Hamiltonian:

$$\mathcal{H} = -\sum_{i,j} J_{ij} \vec{S}_i \vec{S}_j - D \sum_i (S_i^z)^2 - \sum_i \vec{h} \vec{S}_i, \tag{2}$$

where $\vec{S}_i^2 = S^2$ is a classical spin at site i of a simple cubic lattice, exchange constants $J_{ij} = J$ are non zero for nearest neighbors only, and anisotropy is small $J \gg D > 0$.

a) Using Weiss molecular field approach (mean field) find the critical temperature of magnetic ordering phase transition and equations determining spontaneous magnetization at $T < T_c$.

b) Show that this spontaneous magnetization can be obtained at T close to T_c by minimizing the following free energy

$$f = \frac{1}{2\chi_{zz}} m_z^2 + \frac{1}{2\chi_\perp}(m_x^2 + m_y^2) + bm^4, \tag{3}$$

where $m^2 = m_z^2 + m_x^2 + m_y^2$. Express $\chi_{zz}, \chi_\perp, b$ in terms of T_c and D.

Exercise 6: Gradient terms

In RPA approximation calculate the gradient part of free energy for the same model as in the previous exercise. Express the constant g in front of gradient term of free energy in terms of microscopic parameters J, D or T_c.

Hint. Obtain high temperature susceptibility in RPA, write gradient part as $\frac{1}{2} m_a(-q)[\hat{\chi}(q)]^{-1}_{ab} m_b(q)$, and expand $\chi(q)$ at small q.

Exercise 7: Ising ferromagnet

If ferromagnet has a weak easy-axis anisotropy $D \ll J$ one can use Ising model close enough to the phase transition point. How close should temperature be to T_c to use Ising model instead of full anisotropic free energy?

*Exercise 8: Domain wall

Consider the same magnetic system as in exercise 5 at temperature very close but slightly lower than T_c. One can consider *domain wall* – two-dimensional surface separating half-spaces with opposite uniform magnetizations.

a) Minimizing mean field free energy obtained in exercises 5,6 find the "shape" of a domain wall. Namely, find the configuration $m_z(z)$ where z is the axis perpendicular to domain wall and m_z is a component of magnetization along the easy axis. Boundary conditions defining domain wall are, obviously, $m_z(\pm\infty) = \mp m_0$, where m_0 is an equilibrium uniform magnetization. What is the typical width of the domain wall? What is the free energy of domain wall per unit square?

b) What happens if temperature is further away from T_c, namely, in the range where all three components of magnetization play role. To understand it better, find the shape of domain wall at zero temperature assuming that $D \ll J$. What is the typical width of the domain wall? What is the energy of domain wall per unit square?

Hint. Show that in the continuum approximation (2) becomes

$$\mathcal{H} = \int \frac{d^3x}{a^3} \left[Ja^2(\vec{\nabla} S^a)^2 - D(S^z)^2 \right], \tag{4}$$

parameterize spin as $\vec{S} = S(\sin\theta(z), 0, \cos\theta(z))$ and minimize the energy with respect to $\theta(z)$.

*Exercise 9: Spin vortex

Assume now that we have an *easy-plane* ferromagnet described by (2) but with $J \gg -D > 0$. Then spins prefer to lie in the xy plane and there exist another topological defect – spin vortex instead of domain wall. Estimate (very roughly, by analogy with domain wall, no calculations) the size of the core of the vortex. How do you define this core? What happens with magnetization (qualitatively) inside the core? Consider two separate cases: temperature is in the close vicinity of the critical temperature and temperature is zero.