Physics 682: Quantum Magnetism

Problems with stars are not for credit and will NOT be graded.

Homework 3

Exercise 1: Isotropic ferromagnet

Consider a three-dimensional classical isotropic ferromagnet described by a microscopic Hamiltonian:

$$\mathcal{H} = -\sum_{i,j} J_{ij} \vec{S}_i \vec{S}_j, \tag{1}$$

where $\vec{S}_i^2 = S^2$ is a classical spin at site i of a simple cubic lattice, exchange constants $J_{ij} = J$ are non zero for nearest neighbors only. Let us assume that spin $\vec{S} = (S_1, S_2, \ldots, S_N)$ has N-components ($N = 1$, e.g., corresponds to Ising model). Let us assume that the ordering phase transition is described by Landau free energy

$$f = \alpha t m^2 + b m^4 + g(\partial_\mu \vec{m})^2.$$

a) Extract the values of parameters α, b, and g from an RPA approximation to the theory (1). *Hint:* See ex. 5-6 from Homework 2.

b) Write down Ginzburg-Levanyuk criterion of an applicability of the mean field approximation in the vicinity of critical temperature. Use the values of parameters found in a).

c) Is there any small parameter which guarantees the existence of the range where mean field theory works well? Will fluctuation region expand or shrink if one increases (decreases) the dimensionality N of the order parameter?

Exercise 2: Real space renormalization group for one-dimensional Ising model

Consider partition function for one-dimensional Ising model.

$$Z_N(K) = \sum_{\{\sigma_i = \pm 1\}} \exp K \sum_{i=1}^{N} \sigma_i \sigma_{i+1}, \tag{2}$$

where $K = J/T$. Summing over Ising spins on odd sites express this partition function in terms of $Z_{N/2}(\tilde{K})$. Using the found relation between $\tilde{K}$ and K show that renormalization group flow (i.e. repetition of the above procedure) always takes system into the high temperature phase and, therefore, there is no phase transition in 1d Ising model. Using the same relation estimate the correlation length ξ of 1d Ising model at very low temperatures $T \ll J$. Compare the found correlation length with the result of exact solution.

Exercise 3: Classical XY chain

Using "exact" high temperature expansion for 1d classical XY chain

$$Z_N(K) = \int \prod_{i=1}^{N} \frac{d\phi_i}{2\pi} \exp K \sum_{i=1}^{N} \cos(\phi_i - \phi_{i+1}) \tag{3}$$

find the correlation length ξ (defined by $\langle \cos(\phi_x - \phi_{x+r}) \rangle \sim e^{-\frac{r}{\xi}}$) at high $T \gg J$ and low $T \ll J$ temperatures.

Hint: Use identity $e^{K\cos\phi} = \sum_{n=-\infty}^{+\infty} I_n(K) e^{in\phi}$, where $I_n(K)$ is a modified Bessel function.

Exercise 4: Ising model on hexagonal lattice

Derive the exact relation between critical temperatures of ordering of two-dimensional ferromagnetic Ising model on triangular and hexagonal lattice.

Hint. Reproduce Kramers-Wannier duality arguments on hexagonal lattice.

*Exercise 5: Star-triangle relation. Ising model on triangular lattice

Derive "star-triangle" relation

$$\sum_{\sigma=\pm 1} e^{K\sigma(\sigma_1+\sigma_2+\sigma_3)} = a e^{K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)}. \tag{4}$$

Find a and K' in terms of K. Use the star-triangle relation to express the partition function of ferromagnetic Ising model on hexagonal lattice in terms of the partition function on the triangular lattice by summing over Ising spins on one of two sublattices of hexagonal lattice. Use this relation and the one obtained in the exercise 4 to derive an exact expression for ordering temperature of Ising model on triangular (or hexagonal) lattice.

Exercise 6: Mapping of two-dimensional classical XY model onto one-dimensional quantum model

Consider anisotropic limit of classical 2d XY model.

$$Z_N(K) = \int \prod_{i,t} \frac{d\phi_{i,t}}{2\pi} \exp\left[K_t \sum_{i,t} \cos(\phi_{i,t} - \phi_{i,t+1}) + K \sum_{i,t} \cos(\phi_{i,t} - \phi_{i+1,t}) \right], \tag{5}$$

where exchange in "t" (time) direction is much larger than exchange in "space" direction $K \ll 1 \ll K_t$. Go to the limit of continuous time. Perform Wick rotation (replace $t \to it$). The partition function becomes the path integral for some 1d quantum problem. From the obtained Lagrangian of this quantum problem find the Hamiltonian and commutation relations. What is the physical meaning of the obtained quantum problem? We know that 2d XY model has BKT phase transition. What is the nature of this transition in terms of corresponding 1d quantum problem?