23/3/2004. **Due:** Tuesday, April 13, 2004 (in class).

Physics 682: Quantum Magnetism

Problems with stars are not for credit and will NOT be graded.

Homework 4

Exercise 1: 3d XY model

Consider a three dimensional classical XY model on a cubic lattice. Assume that the order-disorder phase transition occurs due to the proliferation of vortex loops. Consider a long single vortex loop of the length $L \gg a$, where a is a lattice constant.

a) Estimate the energy of this loop.

b) Estimate the entropy of this loop.

c) Estimate the temperature of the phase transition as a temperature at which the free energy of a long vortex loop becomes negative.

Exercise 2: Topological defects and textures

What topological defects and textures one should expect in the ordered state of a three-dimensional classical Heisenberg model? What changes if the order parameter is a director instead of a vector? A "director" means a vector without an arrow, i.e., one should identify $\vec{S} \equiv -\vec{S}$. The models with a director as an order parameter are used to describe nematic liquid crystals.

Exercise 3: Independent singlets

Using "addition of representations" for an $SU(2)$ group find how many independent singlets one can construct using

a) 4 spin 1/2

b) 7 spin 1/2

c) 10 spin 1/2

*d) $2N$ spin 1/2 in the limit $N \to \infty$

Exercise 4: Dyson-Maleev transformation

Verify that the operators

$$\begin{aligned} S^+ &= \sqrt{2S} b^\dagger, \\ S^- &= \sqrt{2S}\left(b - \frac{1}{2S} b^\dagger b^2\right), \\ S^z &= -S + b^\dagger b \end{aligned}$$

satisfy standard spin commutation relations if $b, b^\dagger$ are bosonic annihilation and creation operators. What is the value of Casimir operator $\vec{S}^2$ for this representation?

Exercise 5: Spin waves in magnetic field

Using Holstein-Primakoff or Dyson-Maleev transformation, find the spectrum of spin waves (magnons) in a 2d ferromagnet on a triangular lattice in a magnetic field at $T = 0$. The Hamiltonian of a magnet is given by

$$H = -\frac{J}{2} \sum_{i,\delta} \hat{\vec{S}}_i \hat{\vec{S}}_{i+\delta} - h \sum_i \hat{S}_i^z, \tag{1}$$

where i labels the lattice sites of a triangular lattice and δ is a vector connecting a given site to its nearest neighbor. Assume that $S \gg 1$.

Exercise 6: Spin-wave contribution to specific heat

For the magnet in previous problem find the spin-wave contribution to specific heat (per lattice site). Do it in two limits

a) $T \ll h \ll J$

b) $h \ll T \ll J$