15/4/2004. **Due:** Thursday, April 29, 2004 (in class).

Physics 682: Quantum Magnetism

Problems with stars are not for credit and will NOT be graded.

Homework 5

Exercise 1: Spin waves in ferrimagnet (Auerbach 11.3.4)

Consider the ferrimagnet on a cubic lattice in which the spins on sublattice A, of size S_A, couple antiferromagnetically to their neighbors on sublattice B, which have size S_B. Use spin wave theory to compute the dispersions of the elementary excitations.

Exercise 2: Antiferromagnet on hexagonal lattice

Consider a nearest neighbor, Heisenberg antiferromagnet on a planar, hexagonal lattice in a uniform magnetic field h. Using spin wave theory

a) Find the ground state energy of the magnet with first $1/S$ quantum correction and extract from it the magnetic susceptibility $\chi(q)$ at $T = 0$

b) Find the spectrum of magnons at $T = 0$ ($h \neq 0$). What is the physical meaning of two branches of spin waves?

Exercise 3: Topological invariant

Consider a three-dimensional unit vector field $\vec{n} \in S^2$ on a two-dimensional plane $\vec{n}(x, y)$ with constant boundary conditions $\vec{n}(x, y) \to \hat{e}_3$ as $(x, y) \to \infty$. Show that

$$Q = \int d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] \tag{1}$$

is an integer-valued topological invariant. Namely,

a) Show that under small variation $\delta\vec{n}$ of a vector field the corresponding variation $\delta Q = 0$.

b) Show that the integrand in (1) is a Jacobian of the change of variables from x, y to a sphere $\vec{n}$ and it is normalized in such a way that the area of the sphere is 1. Therefore, Q is an integer degree of mapping of a plane (with constant boundary conditions) onto a sphere.

Hint: In b) consider the vicinity of the northern pole of the sphere only and extend your result to the whole sphere by symmetry.

Exercise 4: Bogomol'nyi inequality

Consider the "action" of a two-dimensional O(3) non-linear sigma model

$$S = \frac{1}{2g} \int d^2x \, (\partial_\mu \vec{n})^2. \tag{2}$$

Find the lower bound of this action in a topological sector specified by an invariant Q (see (1)). Namely, consider an obvious inequality

$$\int d^2x \, \left(\partial_\mu \vec{n} \pm \epsilon^{\mu\nu} [\vec{n} \times \partial_\nu \vec{n}]\right)^2 \geq 0, \tag{3}$$

open the square and derive an inequality on S in Q sector.

Exercise 5: Belavin-Polyakov instantons

Let us show that the lower bound found in the previous problem can be achieved. Namely, consider the "self-dual" equation

$$\partial_\mu \vec{n} = -\epsilon^{\mu\nu} [\vec{n} \times \partial_\nu \vec{n}]. \tag{4}$$

We are going to solve this equation in a topological sector Q. Introduce complex coordinates $z = x + iy$, $\bar{z} = x - iy$ and replace $\vec{n}$ by a complex field w (stereographic projection)

$$n_1 + i n_2 = \frac{2w}{1 + |w|^2}, \tag{5}$$

$$n_3 = \frac{1 - |w|^2}{1 + |w|^2}. \tag{6}$$

a) Write down an equation (4) in terms of w and z. What is its the most general solution?

b) Derive an expression for Q in terms of w. What is the value of Q in terms of numbers of zeros and poles of w?

*c) Write down the most general solution of (4) in terms of w for constant boundary conditions and in a topological sector Q.