Exercises to lecture 2

Exercise 1: Bogomol'nyi inequality

Consider the "action" of a two-dimensional O(3) non-linear sigma model

$$S = \frac{1}{2g}\int d^2x\,(\partial_\mu \vec{n})^2. \tag{1}$$

Find the lower bound of this action in a topological sector specified by an invariant Q

$$Q = \int d^2x\,\frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]. \tag{2}$$

Namely, consider an obvious inequality

$$\int d^2x\,\left(\partial_\mu\vec{n}\pm\epsilon^{\mu\nu}[\vec{n}\times\partial_\nu\vec{n}]\right)^2 \geq 0, \tag{3}$$

open the square and derive an inequality on S in Q sector.

Exercise 2: Belavin-Polyakov instantons

Let us show that the lower bound found in the previous problem can be achieved. Namely, consider the "self-dual" equation

$$\partial_\mu\vec{n} = -\epsilon^{\mu\nu}[\vec{n}\times\partial_\nu\vec{n}]. \tag{4}$$

We are going to solve this equation in a topological sector Q. Introduce complex coordinates $z = x+iy$, $\bar{z} = x-iy$ and replace $\vec{n}$ by a complex field w (stereographic projection)

$$n_1 + in_2 = \frac{2w}{1+|w|^2}, \tag{5}$$

$$n_3 = \frac{1-|w|^2}{1+|w|^2}. \tag{6}$$

a) Write down an equation (4) in terms of w and z. What is its the most general solution?

b) Derive an expression for Q in terms of w. What is the value of Q in terms of numbers of zeros and poles of w?

c) Write down the most general solution of (4) in terms of w for constant boundary conditions and in a topological sector Q.

Exercise 3: Topological current for $O(3)$ sigma model in 3 dimensions

The $O(3)$ sigma model is written in terms of the unit vector field $\vec{n}(x)$ ($\vec{n}\in S^2$, or $\vec{n}^2=1$). In three (3+0) dimensions one can form a "topological current"

$$j^\mu = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\vec{n}\cdot\partial_\nu\vec{n}\times\partial_\lambda\vec{n} \tag{7}$$

$$= \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\epsilon_{abc}n^a\partial_\nu n^b\partial_\lambda n^c. \tag{8}$$

Show that for smooth fields $\vec{n}(x)$ we have regardless of equations of motion

$$\partial_\mu j^\mu = 0. \tag{9}$$

What is the meaning of the "conserved topological charge"? (Compare $\int_{t=0} d^2x\, j^0$ with (2)).

Exercise 4: CP^1 representation of $\vec{n}$-field.

The unit vector $\vec{n} \in S^2$ can be represented in terms of complex spinor $z = (z_1, z_2)^T$ as

$$\vec{n} = z^\dagger \vec{\sigma} z \tag{10}$$

or alternatively in the matrix form $\hat{n} \equiv \vec{n} \cdot \vec{\sigma} = 2zz^\dagger - \hat{1}$. The condition $\vec{n}^2 = 1$ requires $z^\dagger z = |z_1|^2 + |z_2|^2 = 1$, i.e., $z \in S^3$. This representation is excessive as $z \to e^{i\alpha} z$ does not change $\vec{n}$. Identifying all points of orbits $e^{i\alpha} z \equiv z$ we have the construction $\vec{n} \in S^2 = S^3/S^1$ with $z \in S^3$ and explicit formula (10). This construction is known as CP^1-representation of an $O(3)$ n-field or as Hopf fibration.

Using (10) express the topological current (7) in terms of z and $z^\dagger$. Express it also in terms of the "gauge field" $a_\mu \equiv z^\dagger i \partial_\mu z$. Notice, that under transformation $z(x) \to e^{i\alpha(x)}$ the gauge field transforms properly as $a_\mu \to a_\mu - \partial_\mu \alpha$. Also notice that the expression for the topological current is gauge invariant (as it is supposed to be as a function of $\vec{n}$ only).

Exercise 5: Topological current and singularities of $\vec{n}$-field.

Let us assume that the $\vec{n}$-field is smooth everywhere on the surface ∂D of the domain D of three-dimensional spacetime. We write down the flux of the topological current through the closed surface ∂D of the domain D.

$$\Phi = \int_{\partial D} j^\mu \, dS_\mu = \int_D \partial_\mu j^\mu \, d^3x. \tag{11}$$

Here $d\vec{S}$ is an area vector directed outwards normal to the surface.

If the $\vec{n}$ field is smooth everywhere inside D the obtained flux is zero because of the topological current conservation. This is not so if there are singularities (defects) of $\vec{n}(x)$ inside the domain D. Explain the geometrical meaning of the flux Φ in this case. Express the flux Φ in terms of singularities inside D. What type of singularities contribute to the flux?

Express Φ in terms of the gauge field a_μ. What is the meaning of Φ in terms of that gauge field? How do singularities inside D look in terms of the gauge field?

Exercise 6: $S^3 \to S^3$ mappings

The mappings of $S^3 \to S^3$ can be divided into homotopy classes labeled by an integer winding number n according to $\pi_3(S^3) = \mathbf{Z}$. In terms of $SU(2)$ matrix valued function $g(x)$ ($SU(2) \sim S^3$) one can write the winding number as

$$n = N \int_{S^3} d^3x \, \epsilon^{\mu\nu\lambda} \, \mathrm{tr} \, \left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g) \right]. \tag{12}$$

Fix the normalization constant N to have any integer number as a possible value for n.

One can parametrize an $SU(2)$ matrix explicitly as $g = \phi^0 + i\vec{\sigma} \cdot \vec{\phi}$ with $(\phi^0)^2 + \vec{\phi}^2 = 1$, i.e., $(\phi^0, \vec{\phi}) \in S^3$. Here $\vec{\sigma}$ is a vector of Pauli matrices. Show that the winding number can also be expressed as ($a, b, c, d = 0, 1, 2, 3$; $\mu, \nu, \lambda = 1, 2, 3$)

$$n = \tilde{N} \int_{S^3} d^3x \, \epsilon^{\mu\nu\lambda} \epsilon_{abcd} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\lambda \phi^d. \tag{13}$$

Fix the normalization constant $\tilde{N}$.

Exercise 7: Pure gauge

Introducing the $SU(2)$ Yang-Mills field (pure gauge)

$$\hat{a}_\mu = a_\mu^a \sigma^a = g^{-1} i \partial_\mu g, \tag{14}$$

derive "zero curvature condition" considering $\partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu$ and using $\partial_\mu g^{-1} = -g^{-1}\partial_\mu g g^{-1}$.

Using the obtained formulas express (12) in terms of the third component of the gauge field a_μ^3 only. Using the parametrization of an $SU(2)$ matrix in terms of $z \in S^3$ as

$$g = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}, \tag{15}$$

show that

$$a_\mu^3 = z^\dagger i \partial_\mu z, \tag{16}$$

i.e. it is the same "gauge field" that we used for CP^1 representation of the n-field. Show that the result obtained for winding number n is gauge invariant with respect to $z \to e^{i\alpha} z$ and, therefore, is the function of the field $\vec{n} = z^\dagger \vec{\sigma} z$ only.

What is the geometrical meaning of the "winding number" n given by (12) for the $\vec{n}$-field? This is a relatively difficult question.