Exercises to lecture 2

Exercise 1: Bogomol’nyi inequality

Consider the “action” of a two-dimensional O(3) non-linear sigma model

\[ S = \frac{1}{2g} \int d^2x (\partial_\mu \vec{n})^2. \quad (1) \]

Find the lower bound of this action in a topological sector specified by an invariant \( Q \)

\[ Q = \int d^2x \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}]. \quad (2) \]

Namely, consider an obvious inequality

\[ \int d^2x (\partial_\mu \vec{n} \pm \epsilon^{\mu\nu} [\vec{n} \times \partial_\nu \vec{n}])^2 \geq 0, \quad (3) \]

open the square and derive an inequality on \( S \) in \( Q \) sector.

Exercise 2: Belavin-Polyakov instantons

Let us show that the lower bound found in the previous problem can be achieved. Namely, consider the “self-dual” equation

\[ \partial_\mu \vec{n} = -\epsilon^{\mu\nu} [\vec{n} \times \partial_\nu \vec{n}]. \quad (4) \]

We are going to solve this equation in a topological sector \( Q \). Introduce complex coordinates \( z = x + iy, \bar{z} = x - iy \) and replace \( \vec{n} \) by a complex field \( w \) (stereographic projection)

\[ n_1 + in_2 = \frac{2w}{1 + |w|^2}, \quad (5) \]
\[ n_3 = \frac{1 - |w|^2}{1 + |w|^2}. \quad (6) \]

a) Write down an equation (4) in terms of \( w \) and \( z \). What is its the most general solution?

b) Derive an expression for \( Q \) in terms of \( w \). What is the value of \( Q \) in terms of numbers of zeros and poles of \( w \)?

c) Write down the most general solution of (4) in terms of \( w \) for constant boundary conditions and in a topological sector \( Q \).

Exercise 3: Topological current for \( O(3) \) sigma model in 3 dimensions

The \( O(3) \) sigma model is written in terms of the unit vector field \( \vec{n}(x) \) \((\vec{n} \in S^2, \text{ or } \vec{n}^2 = 1)\). In three \((3+0)\) dimensions one can form a “topological current”

\[ j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \vec{n} \cdot \partial_\nu \vec{n} \times \partial_\lambda \vec{n} \]
\[ = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} n^a \partial_\nu n^b \partial_\lambda n^c. \quad (7) \]

Show that for smooth fields \( \vec{n}(x) \) we have regardless of equations of motion

\[ \partial_\mu j^\mu = 0. \quad (9) \]

What is the meaning of the “conserved topological charge”? (Compare \( \int_{t=0} d^2x j^0 \) with (2)).
Exercise 4: \( CP^1 \) representation of \( \vec{n} \)-field.

The unit vector \( \vec{n} \in S^2 \) can be represented in terms of complex spinor \( z = (z_1, z_2)^T \) as

\[
\vec{n} = z^\dagger \vec{\sigma} z
\]  

(10)

or alternatively in the matrix form \( \hat{n} \equiv \vec{n} \cdot \vec{\sigma} = 2z z^\dagger - \hat{1} \). The condition \( \vec{n}^2 = 1 \) requires \( z^\dagger z = |z_1|^2 + |z_2|^2 = 1 \), i.e., \( z \in S^3 \). This representation is excessive as \( z \to e^{i\alpha} z \) does not change \( \vec{n} \).

Identifying all points of orbits \( e^{i\alpha} z \equiv z \) we have the construction \( \vec{n} \in S^2 = S^3 / S^1 \) with \( z \in S^3 \) and explicit formula (10). This construction is known as \( CP^1 \)-representation of an \( O(3) \) n-field or as Hopf fibration.

Using (10) express the topological current (7) in terms of \( z \) and \( z^\dagger \). Express it also in terms of the “gauge field” \( a_\mu \equiv z^\dagger i \partial_\mu z \). Notice, that under transformation \( z(x) \to e^{i\alpha(x)} z(x) \) the gauge field transforms properly as \( a_\mu \to a_\mu - \partial_\mu \alpha \). Also notice that the expression for the topological current is gauge invariant (as it is supposed to be as a function of \( \vec{n} \) only).

Exercise 5: Topological current and singularities of \( \vec{n} \)-field.

Let us assume that the \( \vec{n} \)-field is smooth everywhere on the surface \( \partial D \) of the domain \( D \) of three-dimensional spacetime. We write down the flux of the topological current through the closed surface \( \partial D \) of the domain \( D \).

\[
\Phi = \int_{\partial D} j_\mu dS_\mu = \int_D \partial_\mu j^\mu d^3x.
\]  

(11)

Here \( d\vec{S} \) is an area vector directed outwards normal to the surface.

If the \( \vec{n} \) field is smooth everywhere inside \( D \) the obtained flux is zero because of the topological current conservation. This is not so if there are singularities (defects) of \( \vec{n}(x) \) inside the domain \( D \). Explain the geometrical meaning of the flux \( \Phi \) in this case. Express the flux \( \Phi \) in terms of singularities inside \( D \). What type of singularities contribute to the flux?

Express \( \Phi \) in terms of the gauge field \( a_\mu \). What is the meaning of \( \Phi \) in terms of that gauge field? How do singularities inside \( D \) look in terms of the gauge field?

Exercise 6: \( S^3 \to S^3 \) mappings

The mappings of \( S^3 \to S^3 \) can be divided into homotopy classes labeled by an integer winding number \( n \) according to \( \pi_3(S^3) = \mathbb{Z} \). In terms of \( SU(2) \) matrix valued function \( g(x) \) \( (SU(2) \sim S^3) \) one can write the winding number as

\[
n = N \int_{S^3} d^3x \epsilon^{\mu\nu\lambda} \text{tr} \left[ (g^{-1} \partial_\mu g)(g^{-1} \partial_\nu g)(g^{-1} \partial_\lambda g) \right].
\]  

(12)

Fix the normalization constant \( N \) to have any integer number as a possible value for \( n \).

One can parametrize an \( SU(2) \) matrix explicitly as \( g = \phi^0 + i\vec{\phi} \cdot \vec{\sigma} \) with \( (\phi^0)^2 + \vec{\phi}^2 = 1 \), i.e., \( (\phi^0, \vec{\phi}) \in S^3 \). Here \( \vec{\sigma} \) is a vector of Pauli matrices. Show that the winding number can also be expressed as \( (a, b, c, d = 0, 1, 2, 3; \mu, \nu, \lambda = 1, 2, 3) \)

\[
n = \tilde{N} \int_{S^3} d^3x \epsilon^{\mu\nu\lambda} \epsilon_{abcd} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\lambda \phi^d.
\]  

(13)

Fix the normalization constant \( \tilde{N} \).
Exercise 7: Pure gauge

Introducing the SU(2) Yang-Mills field (pure gauge)

$$\hat{a}_\mu = a_\mu^a \sigma^a = g^{-1} i \partial_\mu g,$$  \hspace{1cm} (14)

derive “zero curvature condition” considering $\partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu$ and using $\partial_\mu g^{-1} = -g^{-1} \partial_\mu g g^{-1}$.

Using the obtained formulas express (12) in terms of the third component of the gauge field $a_3^\mu$ only. Using the parametrization of an SU(2) matrix in terms of $z \in S^3$ as

$$g = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix},$$  \hspace{1cm} (15)

show that

$$a_3^\mu = z^\dagger i \partial_\mu z,$$  \hspace{1cm} (16)

i.e. it is the same “gauge field” that we used for CP$^1$ representation of the $n$-field. Show that the result obtained for winding number $n$ is gauge invariant with respect to $z \rightarrow e^{i\alpha} z$ and, therefore, is the function of the field $\vec{n} = z^\dagger \vec{\sigma} z$ only.

What is the geometrical meaning of the “winding number” $n$ given by (12) for the $\vec{n}$-field? This is a relatively difficult question.