Exercises to lecture 3

The details of calculation of fermionic determinants can be found in Ref.[1].

Exercise 1: Boundary spin 1/2 states of a Haldane's chain

Consider the "action" of a two-dimensional O(3) non-linear sigma model with topological term

$$\begin{aligned} S &= S_{NLSM} + S_\theta, & (1)\\ S_{NLSM} &= \frac{1}{2g}\int d^2x\,(\partial_\mu\vec{n})^2, & (2)\\ S_\theta &= i\theta Q, & (3)\\ Q &= \int d^2x\,\frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}[\partial_\mu\vec{n}\times\partial_\nu\vec{n}]. & (4)\end{aligned}$$

This action can be derived as a continuum limit of Heisenberg spin chain with large spins on sites. In the latter case $g = 2/S$ and $\theta = 2\pi S$. In the case of integer S the spin chain is massive and there are no bulk excitations at low energies (smaller than the gap).

Let us assume that the action (1) is defined on the open chain of the length L. Show that the topological theta-term formally defined on the open chain reduces to two WZW (0+1-dimensional) terms at the boundary of spacetime, i.e. at the ends of the spin chain. This means that we expect two quantum spins living at the ends of the spin chain. Show that the coefficient in front corresponds to the value of those spins $S/2$. In particular, it means that the boundary states of $S = 1$ spin chain correspond to spin-1/2!

Remark: Neglecting the NLSM part of the action is possible in this exercise only because of the gap in the bulk at the integer value of spin.

Exercise 2: Theta term by reduction from WZW term

Consider the two-dimensional WZW term defined as an integral over three-dimensional ball D^3

$$S_{WZW} = ik\frac{1}{12\pi}\int_{D^3} d^3x\,\epsilon^{\mu\nu\lambda}\,\mathrm{tr}\,\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g)\right], \tag{5}$$

where $g \in SU(2)$ is a matrix-valued field. This field is defined on the two-dimensional spacetime $S^2 = \partial D^3$ and smoothly extended to the interior of the ball from the boundary (actual spacetime).

a) Show that (5) is well defined if the coupling constant k is integer. Namely, its value does not depend on the way the field g is extended from S^2 to D^3.

b) Let us make a reduction and substitute in (5) $g = \cos\alpha + i\sin\alpha\,\vec{n}\cdot\vec{\sigma}$ with $\alpha = const$ and $\vec{n} \in S^2$ a unit vector. Show by explicit calculation that (5) reduces to the theta-term made out of $\vec{n}$ and given by the integral over spacetime S^2. Find the corresponding value of θ coupling in terms of α and k.

Exercise 3: WZW term from the fermionic determinant in two dimensions

Let us consider two-dimensional fermions coupled to NLSM given in terms of the four component unit vector $(\pi_0, \vec{\pi}) \in S^3$ (i.e., $\pi_0^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$). The Euclidian Lagrangian is given by

$$\mathcal{L}_2 = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) + im(\pi_0 + i\gamma^5\vec{\pi}\cdot\vec{\tau})\right]\psi, \tag{6}$$

where $\mu = 1, 2$ is a spacetime index, $\vec{\tau}$ is a vector of Pauli matrices, and A_μ is an external gauge field probing fermionic currents.

We assume that the bosonic field π changes slowly on the scale of the "mass" m. Then one can integrate out fermionic degrees of freedom and obtain an induced effective action for the π-field as a functional determinant.

$$S_{eff} = -\log \mathrm{Det}\, D, \tag{7}$$
$$D = i\gamma^\mu(\partial_\mu - iA_\mu) + im(\pi_0 + i\gamma^5 \vec{\pi}\cdot\vec{\tau}). \tag{8}$$

We calculate the effective action using the gradient expansion method. Namely, we calculate the variation of (7) with respect to the π-field and use

$$\delta S_{eff} = -\delta \log \mathrm{Det}\, D = -\mathrm{Tr}\, \delta \log D = -\mathrm{Tr}\, \delta D\, D^{-1} = -\mathrm{Tr}\, \delta D\, D^\dagger (DD^\dagger)^{-1}. \tag{9}$$

a) Calculate $DD^\dagger$ for (8). Observe that this object depends only on gradients of π-field.

b) Expand $(DD^\dagger)^{-1}$ in those gradients. This will be the expansion in $1/m$. (It is convenient to introduce notation $G_0^{-1} = -\partial_\mu^2 + m^2$).

c) Calculate functional traces of the terms up to the order of m^0. Use the plane wave basis to calculate the trace $\mathrm{Tr}\,(\hat{X}) \to \int d^2x \int \frac{d^2p}{(2\pi)^2} e^{-i\vec{p}\cdot\vec{x}} \hat{X} e^{i\vec{p}\cdot\vec{x}}$.

d) Identify the variation of the topological term in the obtained expression. It contains the antisymmetric tensor $\epsilon^{\mu\nu}$ and is proportional to $\mathrm{sgn}\,(m)$.

e) Remove the variation from the obtained expression and find S_{eff} up to the m^0 order. Remember that the removal of the variation for topological term requires Wess-Zumino trick (introduce an auxiliary ball D^3 with the spacetime being the boundary of the ball, extend fields to the ball, etc...).

Exercise 4: Topological current term from the fermionic determinant in three dimensions

Let us consider three-dimensional fermions coupled to NLSM given in terms of the three component unit vector $\vec{n} \in S^3$ (i.e., $n_1^2 + n_2^2 + n_3^2 = 1$). The Euclidian Lagrangian is given by

$$\mathcal{L}_3 = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) + im\vec{n}\cdot\vec{\tau}\right]\psi, \tag{10}$$

where $\mu = 1, 2, 3$ is a spacetime index, $\vec{\tau}$ is a vector of Pauli matrices, and A_μ is an external gauge field probing fermionic currents.

We assume that the bosonic field π changes slowly on the scale of the "mass" m. Then one can integrate out fermionic degrees of freedom and obtain an induced effective action for the π-field as a functional determinant.

$$S_{eff} = -\log \mathrm{Det}\, D, \tag{11}$$
$$D = i\gamma^\mu(\partial_\mu - iA_\mu) + im\vec{n}\cdot\vec{\tau}. \tag{12}$$

Calculate the variation of (11) with respect to the external gauge field A_μ up to the order of m^0 and derive the expression for the topological current. What is the (physical) geometrical meaning of the topological current term?

References

[1] A. G. Abanov and P. B. Wiegmann, Nucl. Phys. **B 570**, 685-698 (2000).
Theta-terms in nonlinear sigma models