Exercises to lecture 4

Exercise 1: WZW in $0+1$, preliminaries

Consider a three-dimensional unit vector field $\vec{n}(x,y)$ ($\vec{n} \in S^2$) defined on a two-dimensional disk D. Define

$$W_0 = \int_D d^2x \, \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n}[\partial_\mu \vec{n} \times \partial_\nu \vec{n}] = \int_D \frac{1}{16\pi i} \mathrm{tr}\,[\hat{n} d\hat{n} d\hat{n}], \qquad (1)$$

where the latter expression is written in terms of differential forms and $\hat{n} = \vec{n} \cdot \vec{\sigma}$.

a) Calculate the variation of W_0 with respect to $\vec{n}$. Show that the integral becomes the integral over disk D of the complete divergence (of the exact form).

b) Parametrize the boundary ∂D of the disk by parameter t, apply Gauss-Stokes theorem and express the result of the variation using only the values of $\vec{n}(t)$ at the boundary.

We showed that the variation of W depends only on the boundary values of $\vec{n}$-field.

Exercise 2: WZW in $0+1$, definition

Assume that we are given the time evolution of $\vec{n}(t)$ field ($\vec{n} \in S^2$). We also assume that time can be compactified, i.e. $\vec{n}(t=\beta) = \vec{n}(t=0)$. Consider the two-dimensional disk D which boundary ∂D is parametrized by time $t \in [0, \beta]$. The WZW term is defined by

$$S_{WZW} = i4\pi S \, W_0[\vec{n}], \qquad (2)$$

where S is some constant, W_0 is given by Eq. (1), and $\vec{n}(x,y)$ is some arbitrary smooth extension of $\vec{n}(t)$ from the boundary to an interior of the disk.

Let us show that the WZW term is well defined and (almost) does not depend on the extension of $\vec{n}(t)$ to the interior of D.

Consider two different extensions $\vec{n}^{(1)}(x,y)$ and $\vec{n}^{(2)}(x,y)$ of the same $\vec{n}(t)$ and corresponding values $W_0^{(1)}$ and $W_0^{(2)}$ of the functional W_0. Show that the difference $W_0^{(1)} - W^{(2)}$ is an integer number - the degree Q of mapping $S^2 \to S^2$. The second S^2 here is a target space of $\vec{n}$. How did the first S^2 appear?

We see that $S_{WZW}[\vec{n}(t)]$ is a multi-valued functional which depends on the extension of $\vec{n}$ to the disk D. However, the weight in partition function is given by $e^{-S_{WZW}}$ and can be made single-valued functional if the coupling constant S is "quantized". Namely, if $2S \in \mathbf{Z}$ (S - half-integer number) the $e^{-S_{WZW}}$ is a well-defined single-valued functional.

Exercise 3: WZW in $0+1$, spin precession

Let us consider the quantum-mechanical action of the unit vector $\vec{n}(t)$ with the (Euclidian) action

$$S_h = S_{WZW}[\vec{n}(t)] - S \int dt \, \vec{h} \cdot \vec{n}(t), \qquad (3)$$

where W is given by (1) and $\vec{h}$ is some constant three-component vector (magnetic field).

Find the classical equation of motion for $\vec{n}(t)$ from the variational principle $\delta S_h = 0$. Remember that one has a constraint $\vec{n}^2 = 1$ which can be taken into account using, e.g., Lagrange multiplier trick.

The obtained expression is the equation of spin precession and S_{WZW} is a proper, explicitly $SU(2)$ invariant action for the free spin S.

Exercise 4: WZW in $0+1$, quantization

Show that the classical equations of motion obtained from S_h correspond to Heisenberg equations (in real time) $\partial_t \hat{\vec{S}} = i\left[H, \hat{\vec{S}}\right]$ for the quantum spin operator $\hat{\vec{S}}$

$$\left[S^a, S^b\right] = i\epsilon^{abc} S^c \tag{4}$$

obtained from the Hamiltonian of a spin in magnetic field

$$H = -\vec{h} \cdot \hat{\vec{S}}. \tag{5}$$

Obtain the commutation relations of quantum spin (4) from the topological part S_{WZW}. Notice that this topological action is linear in time derivative and, therefore, does not contribute to the Hamiltonian. Nevertheless, it defines commutation relations between components of the spin operator.

Hint: You can either use local coordinate representation of the unit vector in terms of spherical angles $\vec{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ or use the general formalism of obtaining Poisson bracket from the symplectic form given in S_{WZW}.

Exercise 5: Reduction of WZW to the theta-term in $0+1$

Let us assume that the field $\vec{n}(t)$ is constrained so that it takes values on a circle given in spherical coordinates by $\theta = \theta_0 = const$. Find the value of the topological term S_{WZW} on such configurations (notice that this constraint is not applicable in the interior of the disk D, only at its physical boundary). Show that the obtained topological term is a theta-term in $0+1$ corresponding to $S^1 \to S^1$.

What is the value of the coefficient in front of that topological term? What is the value of corresponding "magnetic flux" through a ring? For $S = 1/2$ which reduction (value of θ_0) corresponds to the half of the flux quantum?

Exercise 6: WZW in $0+1$, derivation from fermions

Consider an Euclidian action of a fermion coupled to a unit vector

$$S_E = \int d\tau\, \psi^\dagger D \psi, \tag{6}$$

where

$$D = \partial_\tau - m\vec{n} \cdot \vec{\tau} \tag{7}$$

with $\vec{n} \in S^2$ and $\vec{\tau}$ the vector of Pauli matrices. We obtain an effective action for $\vec{n}$ induced by fermions as

$$e^{-S_{eff}} = \int D\psi D\psi^\dagger e^{-S_E} = \mathrm{Det}\, D \tag{8}$$

or

$$S_{eff} = -\log \mathrm{Det}\, D = -\mathrm{Tr}\, \log D. \tag{9}$$

We calculate the variation of S_{eff} with respect to $\vec{n}$ as

$$\delta S_{eff} \quad = \quad -\mathrm{Tr}\,\delta D D^{-1} = -\mathrm{Tr}\,\delta D D^{\dagger}(DD^{\dagger})^{-1}, \tag{10}$$

where $D^{\dagger} = -\partial_{\tau} - -m\vec{n}\cdot\vec{\tau}$. We have

$$DD^{\dagger} \quad = \quad -\partial_{\tau}^{2} + m^{2} - m\dot{\vec{n}}\cdot\vec{\tau} = G_0^{-1} - m\dot{\vec{n}}\cdot\vec{\tau}. \tag{11}$$

Expand (10) in $1/m$ up to the term m^0 and calculate functional traces. Show that the term of the order m^0 is a variation of the WZW term in 0+1 dimensions. Restore S_{eff} from its variation. What is the coefficient in front of the WZW term? To what value of spin does it correspond?

Exercise 7: Haldane's theta-term from WZW in $0+1$

Following Haldane, we consider the chain of quantum spins described by the action

$$S_{chain} = \sum_{j} i4\pi S\, W_0[\vec{n}_j(t)], \tag{12}$$

where the sum is taken over the sites of a 1d spin chain labeled by j. Let us assume that as a result of the dynamics we have an antiferromagnetic chain and that the field $(-1)^j\vec{n}_j$ is a smooth one and can be approximated by the smooth field $\vec{n}(x,t)$, where $x = ja$ and a is a lattice constant. Show that (12) reduces to the theta term in $2+0$ dimensions corresponding to the mapping $S^2 \to S^2$. What is the value of the coefficient θ in front of that term?