Exercises to lecture 4

Exercise 1: WZW in $0 + 1$, preliminaries

Consider a three-dimensional unit vector field $\vec{n}(x,y)$ ($\vec{n} \in S^2$) defined on a two-dimensional disk $D$. Define

$$W_0 = \int_D d^2x \frac{1}{8\pi} \epsilon^{\mu\nu} \vec{n} \partial_\mu \vec{n} \times \partial_\nu \vec{n} = \int_D \frac{1}{16\pi i} \text{tr} [\hat{n} d\vec{n} d\vec{n}], \quad (1)$$

where the latter expression is written in terms of differential forms and $\hat{n} = \vec{n} \cdot \vec{\sigma}$.

a) Calculate the variation of $W_0$ with respect to $\vec{n}$. Show that the integral becomes the integral over disk $D$ of the complete divergence (of the exact form).

b) Parametrize the boundary $\partial D$ of the disk by parameter $t$, apply Gauss-Stokes theorem and express the result of the variation using only the values of $\vec{n}(t)$ at the boundary.

We showed that the variation of $W$ depends only on the boundary values of $\vec{n}$-field.

Exercise 2: WZW in $0 + 1$, definition

Assume that we are given the time evolution of $\vec{n}(t)$ field ($\vec{n} \in S^2$). We also assume that time can be compactified, i.e. $\vec{n}(t = \beta) = \vec{n}(t = 0)$. Consider the two-dimensional disk $D$ which boundary $\partial D$ is parametrized by time $t \in [0, \beta]$. The WZW term is defined by

$$S_{WZW} = i4\pi S W_0[\vec{n}], \quad (2)$$

where $S$ is some constant, $W_0$ is given by Eq. (1), and $\vec{n}(x,y)$ is some arbitrary smooth extension of $\vec{n}(t)$ from the boundary to an interior of the disk.

Let us show that the WZW term is well defined and (almost) does not depend on the extension of $\vec{n}(t)$ to the interior of the disk.

Consider two different extensions $\vec{n}^{(1)}(x,y)$ and $\vec{n}^{(2)}(x,y)$ of the same $\vec{n}(t)$ and corresponding values $W_0^{(1)}$ and $W_0^{(2)}$ of the functional $W_0$. Show that the difference $W_0^{(1)} - W_0^{(2)}$ is an integer number - the degree $Q$ of mapping $S^2 \rightarrow S^2$. The second $S^2$ here is a target space of $\vec{n}$. How did the first $S^2$ appear?

We see that $S_{WZW}[\vec{n}(t)]$ is a multi-valued functional which depends on the extension of $\vec{n}$ to the disk $D$. However, the weight in partition function is given by $e^{-S_{WZW}}$ and can be made single-valued functional if the coupling constant $S$ is "quantized". Namely, if $2S \in \mathbb{Z}$ ($S$ - half-integer number) the $e^{-S_{WZW}}$ is a well-defined single-valued functional.

Exercise 3: WZW in $0 + 1$, spin precession

Let us consider the quantum-mechanical action of the unit vector $\vec{n}(t)$ with the (Euclidian) action

$$S_h = S_{WZW}[\vec{n}(t)] - S \int dt \vec{h} \cdot \vec{n}(t), \quad (3)$$

where $W$ is given by (1) and $\vec{h}$ is some constant three-component vector (magnetic field).

Find the classical equation of motion for $\vec{n}(t)$ from the variational principle $\delta S_h = 0$. Remember that one has a constraint $\vec{n}^2 = 1$ which can be taken into account using, e.g., Lagrange multiplier trick.
The obtained expression is the equation of spin precession and $S_{WZW}$ is a proper, explicitly $SU(2)$ invariant action for the free spin $S$.

**Exercise 4: WZW in $0 + 1$, quantization**

Show that the classical equations of motion obtained from $S_h$ correspond to Heisenberg equations (in real time) $\partial_t \hat{S} = i \left[H, \hat{S}\right]$ for the quantum spin operator $\hat{S}$

$$[S^a, S^b] = i \epsilon^{abc} S^c$$

obtained from the Hamiltonian of a spin in magnetic field

$$H = -\vec{h} \cdot \hat{S}.$$  

Obtain the commutation relations of quantum spin (4) from the topological part $S_{WZW}$. Notice that this topological action is linear in time derivative and, therefore, does not contribute to the Hamiltonian. Nevertheless, it defines commutation relations between components of the spin operator.

*Hint:* You can either use local coordinate representation of the unit vector in terms of spherical angles $\vec{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ or use the general formalism of obtaining Poisson bracket from the symplectic form given in $S_{WZW}$.

**Exercise 5: Reduction of WZW to the theta-term in $0 + 1$**

Let us assume that the field $\vec{n}(t)$ is constrained so that it takes values on a circle given in spherical coordinates by $\theta = \theta_0 = \text{const}$. Find the value of the topological term $S_{WZW}$ on such configurations (notice that this constraint is not applicable in the interior of the disk $D$, only at its physical boundary). Show that the obtained topological term is a theta-term in $0 + 1$ corresponding to $S^1 \rightarrow S^1$.

What is the value of the coefficient in front of that topological term? What is the value of corresponding “magnetic flux” through a ring? For $S = 1/2$ which reduction (value of $\theta_0$) corresponds to the half of the flux quantum?

**Exercise 6: WZW in $0 + 1$, derivation from fermions**

Consider an Euclidian action of a fermion coupled to a unit vector

$$S_{E} = \int d\tau \, \psi^\dagger D\psi,$$

where

$$D = \partial_\tau - m \vec{n} \cdot \vec{\tau}$$

with $\vec{n} \in S^2$ and $\vec{\tau}$ the vector of Pauli matrices. We obtain an effective action for $\vec{n}$ induced by fermions as

$$e^{-S_{eff}} = \int D\psi D\psi^\dagger e^{-S_{E}} = \text{Det} D$$

or

$$S_{eff} = -\log \text{Det} D = -\text{Tr} \log D.$$
We calculate the variation of $S_{\text{eff}}$ with respect to $\vec{n}$ as
\[ \delta S_{\text{eff}} = -\text{Tr} \delta D D^{-1} = -\text{Tr} \delta D D^\dagger (D D^\dagger)^{-1}, \] (10)
where $D^\dagger = -\partial_x - m\vec{n} \cdot \vec{\tau}$. We have
\[ DD^\dagger = -\partial_x^2 + m^2 - m \vec{n} \cdot \vec{\tau} = G_0^{-1} - m \vec{n} \cdot \vec{\tau}. \] (11)

Expand (10) in $1/m$ up to the term $m^0$ and calculate functional traces. Show that the term of the order $m^0$ is a variation of the WZW term in $0+1$ dimensions. Restore $S_{\text{eff}}$ from its variation. What is the coefficient in front of the WZW term? To what value of spin does it correspond?

**Exercise 7: Haldane's theta-term from WZW in $0 + 1$**

Following Haldane, we consider the chain of quantum spins described by the action
\[ S_{\text{chain}} = \sum_j i4\pi S W_0[\vec{n}_j(t)], \] (12)
where the sum is taken over the sites of a $1d$ spin chain labeled by $j$. Let us assume that as a result of the dynamics we have an antiferromagnetic chain and that the field $(-1)^j \vec{n}_j$ is a smooth one and can be approximated by the smooth field $\vec{n}(x,t)$, where $x = ja$ and $a$ is a lattice constant. Show that (12) reduces to the theta term in $2 + 0$ dimensions corresponding to the mapping $S^2 \rightarrow S^2$. What is the value of the coefficient $\theta$ in front of that term?