2/15/2008. **Due:** Friday, Feb. 29, 2008 (in class).

Physics 556: Solid State Physics II

Homework 2

Read: **FW** 6, 7, 8, 9

"**FW**" refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1

Consider a system of fermions or bosons, created by the field $\psi^\dagger(x)$ interacting under the pair potential $V(r) = U\,\theta(R-r)$, where r is the distance between particles, R some parameter, and $\theta(x)$ is a step function.

a) Write the interaction in second quantized form.

b) Switch to the momentum basis, where $\psi(x) = \int \frac{d^3k}{(2\pi)^2}\, c_k e^{ikx}$. Verify that $[c_k, c_{k'}^\dagger]_\pm = (2\pi)^3\delta^{(3)}(k-k')$ and write the interaction in this new basis. Please, sketch the form of the interaction in momentum space.

*Exercise 2: Going around the pole

Express the number density of the Fermi gas in terms of the Green's function $G_{\alpha\beta}(\epsilon, p)$ starting from

$$n(x) = \pm i \lim_{t'\to t^+, x'\to x} \operatorname{tr} G(x,t;x',t').$$

Calculate the integral and obtain the formula $k_F^3 = 3\pi^2 n$ for the Fermi wavevector k_F.

Exercise 3: Density in momentum space

a) Express the density of particles in momentum space $n_p = \langle \psi_p^\dagger \psi_p \rangle$ in terms of the Green's function $G_{\alpha\beta}(\epsilon, p)$.

b) Calculate n_p for non-interacting Fermi gas using the explicit form

$$G_{\alpha\beta}(\epsilon, p) = \frac{\delta_{\alpha\beta}}{\epsilon - \xi(p) + i0\,\mathrm{sgn}\,(\epsilon)},$$

where $\xi(p) \equiv \epsilon(p) - \mu$ for particles with the dispersion $\epsilon(p)$ $(= \frac{p^2}{2m})$.

Exercise 4: Fermions on a line (exact)

Find the exact Green's function $G_{\alpha\beta}(\epsilon, x, x')$ for noninteracting spinless Fermi particles on the line.

Exercise 5: Fermions on a line (approximate)

Find the Green's function $G_{\alpha\beta}(\epsilon, x, x')$ for noninteracting spinless Fermi particles on the line linearizing the spectrum around "Fermi surface". Compare the result with the result of the exact calculation.

Exercise 6: Fermions on a half-line

Find the Green's function $G_{\alpha\beta}(\epsilon, x, x')$ for noninteracting spinless Fermi particles on the half-line $x > 0$ with boundary conditions $\psi|_{x=0} = 0$ (impenetrable hard wall).

Hint: Use the method of images.

*Exercise 7: Friedel oscillations

Show that the density of fermions on a half-line $x > 0$ with impenetrable hard wall boundary conditions at $x = 0$ oscillates as a function of x. What is the period of these oscillations?

Exercise 8: Green's function in coordinate space

Find the Green's function $G(\epsilon, \mathbf{r}_1 - \mathbf{r}_2)$ of a non-interacting Fermi gas at $|\mathbf{r}_1 - \mathbf{r}_2|k_F \gg 1$ using two methods:

a) integrating over ξ from $-\infty$ to $+\infty$.
b) integrating exactly over d^3p.
c) Compare the results.

Hint: Integrate over the angles first.

Exercise 9

Let us assume that we know exact eigenfunctions $\psi_n(x)$ and exact eigenenergies E_n of a single-particle problem in some external potential. Express the many body Green's function $G(E; x, x')$ in terms of these eigenenergies and eigenfunctions. Assume that particles are non-interacting fermions and that the chemical potential is μ.

Exercise 10

Consider a non-interacting Fermi gas (with $s = 1/2$) in the presence of an external potential $V_{\alpha\beta}(x)$ (this potential can flip spin). We write the interaction with the potential in the second quantized form as

$$\hat{H}_1 = \int d^3x\, \psi^\dagger_\alpha(x) V_{\alpha\beta}(x) \psi_\beta(x),$$

where summation over spin indices is assumed.

a) Write down the time derivative of an exact single-particle Green's function

$$i\hbar \frac{\partial}{\partial t} G_{\alpha\beta}(x, t; x', t')$$

and obtain the closed partial differential equation for G using operator equations of motion for $\psi(x, t)$.

*b) Write down the integral equation for G using the Green's function G_0 of a non-interacting Fermi gas without an external potential.

*c) Write down this integral equation in terms of diagrams.