2/29/2008. **Due:** Monday, March. 10, 2008 (in class).

Physics 556: Solid State Physics II

Homework 3

Read: **FW** 6, 7, 8, 9

"**FW**" refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1: Discontinuity of the momentum distribution

The momentum distribution of particles (say, with spin up) is

$$n_{k\uparrow} = \langle \psi^\dagger_{k\uparrow}\psi_{k\uparrow}\rangle = -2i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,G(k,\omega)e^{i\eta\omega}, \tag{1}$$

where as usual the limit $\eta \to +0$ is assumed.

Assume that the Green's function of interacting fermions is given by

$$G(k,\omega) = \frac{a}{\omega - \epsilon_k/\hbar + i\gamma_k\,\mathrm{sgn}\,(k-k_F)} + G_{\mathrm{reg}}(k,\omega), \tag{2}$$

where a is some number and $\gamma_k > 0$. Here $G_{\mathrm{reg}}(k,\omega)$ is the part of the Green's function which is less singular than the "leading" pole contribution.

Find the contribution of the pole part of (2) to the momentum distribution (1). Can you make any conclusions about possible values of the constant a?

Exercise 2: The Hadamard lemma and fermionic bilinears

One of the most useful relations in quantum field theory is

$$e^{iS}Oe^{-iS} = O + i[S,O] + \frac{i^2}{2!}[S,[S,O]] + \frac{i^3}{3!}[S,[S,[S,O]]] + \ldots, \tag{3}$$

where S and O are some operators.

a) Prove this relation expanding $e^{i\lambda S}Oe^{-i\lambda S}$ in Taylor series in λ at $\lambda = 1$.
Hint: the coefficient of the n-th order term of Taylor expansion of $f(\lambda)$ is $f^{(n)}(1)/n!$.

b) Suppose $S = \psi^\dagger A\psi \equiv \psi^\dagger_i A_{ij}\psi_j$ is quadratic and $O = (a\psi) \equiv a_i\psi_i$ is linear in ψ and ψ_i are canonic fermions $\{\psi_i, \psi^\dagger_j\} = \delta_{ij}$. Summation over repeated indices is assumed. Calculate (express in terms of the matrix A and vector a) the following expression

$$e^{i\psi^\dagger A\psi}(a\psi)e^{-i\psi^\dagger A\psi}. \tag{4}$$

Exercise 3: example of Wick's theorem (FW 3.9)

Make the canonical transformation to particles and holes for fermions $c_{k\lambda} = \theta(k-k_F)a_{k\lambda} + \theta(k_F - k)b^\dagger_{-k\lambda}$. By applying Wick's theorem, prove the relation

$$\begin{aligned} c^\dagger_1 c^\dagger_2 c_4 c_3 &= N(c^\dagger_1 c^\dagger_2 c_4 c_3) + \theta(k_F - k_2)\left[\delta_{24}N(c^\dagger_1 c_3) - \delta_{23}N(c^\dagger_1 c_4)\right] \\ &+ \theta(k_F - k_1)\left[\delta_{13}N(c^\dagger_2 c_4) - \delta_{14}N(c^\dagger_2 c_3)\right] + \theta(k_F-k_1)\theta(k_F-k_2)\left[\delta_{13}\delta_{24} - \delta_{14}\delta_{23}\right], \end{aligned}$$

where the normal-ordered products on the right hand side now refer to the new particle and hole operators, and the subscripts indicate the quantum numbers (k, λ).

Exercise 4: Multiparticle expectation value

Consider the following expectation value

$$P_N = \left\langle \prod_{j=1}^{N} \psi_j \psi_j^\dagger \right\rangle, \tag{5}$$

where ψ_j is an annihilation operator of a one-dimensional spinless lattice fermion at the lattice site labeled by an integer j. All operators in (5) are taken at the same time. Using the Wick's theorem express P_N in terms of single particle averages $G_{ij} \equiv \langle \psi_i \psi_j^\dagger \rangle$. What is the physical meaning of P_N?

*Exercise 5: Fluctuations of the number of 1d fermions

For the gas of spinless one-dimensional fermions write down the operator of the number of particles $\hat{N}_L$ on the interval $0 < x < L$ in second quantization form. For large $L \ll k_F^{-1}$ obtain the formula

$$\langle \delta N_L^2 \rangle = \langle N_L^2 \rangle - \langle N_L \rangle^2 = \frac{1}{\pi^2} \ln k_F L. \tag{6}$$

Exercise 6: Friedel oscillations

Consider a non-interacting Fermi gas in the presence of an impurity - a delta-functional potential $U\delta^{(3)}(x)$. The Hamiltonian of the perturbation can be written as

$$\hat{H}_1(t) = U\psi^\dagger(0,t)\psi(0,t). \tag{7}$$

Assume that U is small and calculate the density of fermions as a function of r the distance to the impurity for $r \ll k_F^{-1}$ in the first order in U. Show that the density of fermions oscillates as a function of r. What is the period of these oscillations?

Hint: Use the Green's function in coordinate representation obtained in HW2, ex 8a.

Exercise 7: Ruderman-Kittel-Kasuya-Yosida (RKKY) oscillations

Consider a non-interacting electron gas in the presence of a localized spin $\vec{S}$, interacting with the local spin density of electrons. The interaction can be written as

$$\hat{H}_1(t) = J\vec{S}\psi_\alpha^\dagger(0,t)\vec{\sigma}_{\alpha\beta}\psi_\beta(0,t), \tag{8}$$

where $\vec{\sigma}$ is a vector made of Pauli matrices and summation over spin indices α, β is assumed. Assume that J is small and calculate the spin density of electrons as a function of r the distance to the spin for $r \ll k_F^{-1}$ in the first order in J. Show that the spin density of electrons oscillates as a function of r. What is the period of these oscillations?

Hint: see the previous problem

Exercise 8: Scattering amplitude in quantum mechanics

Consider a single particle in the presence of an external potential $V(x)$. The Green's function of the particle satisfies the integral equation

$$G = G_0 + G_0 V G, \tag{9}$$

with

$$G_0(\epsilon, p) = \frac{1}{\epsilon - p^2/2m + i\eta}. \tag{10}$$

Notice that for a single particle $k_F = 0$ and the sign of the imaginary part in the denominator is always positive (the function is analytic in the upper half plane).

It is convenient to introduce the scattering amplitude (t-matrix) so that

$$G = G_0 + G_0 t G_0, \tag{11}$$

where multiplication here is understood as an operator multiplication.

a) Write down the integral equation for t-matrix in terms of G_0 and V. Write all three forms of the same equation: symbolic (similar to (9)), diagrammatic (denote t as a shaded box), and explicit. For the latter one use notation $t_{k,k'}(\epsilon)$ and $G_0(k,\epsilon)\delta_{k,k'}$. Notice that t is not diagonal in momentum (depends on both k and k') because of the absence of the translational invariance in the presence of the potential $V(x)$.

b) Evaluate the t-matrix for a delta-function potential $V(x) = U\delta^{(d)}(x)$ ($d = 1, 2, 3, \ldots$ is a dimension of space).

Hint: the t-matrix is momentum independent.

c) Show that in dimensions $d \leq 2$, a bound-state forms for arbitrarily weak scattering potential.

Hint: bound state means the negative energy pole in the t matrix.