Physics 556: Solid State Physics II

Homework 5

Read: FW 10, 11, 12

"**FW**" refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1: Outgoing Green's function and Yukawa potential

a) Calculate the integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2 - k^2 \pm i\eta}.$$

b) Calculate the integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2 + m^2}$$

Exercise 2: Exchange enhancement of susceptibility

Consider a Fermi gas with contact interaction between particles $V(x - x') = g\delta^{(3)}(x - x')$ in a uniform external magnetic field B at $T = 0$. Keeping only Zeeman part of an interaction with magnetic field (and neglecting orbital effects of magnetic field) we write the interaction with magnetic field as

$$H_B = \omega_B \sum_p (c^\dagger_{p,\uparrow} c_{p,\uparrow} - c^\dagger_{p,\downarrow} c_{p,\downarrow}), \tag{1}$$

where $\omega_B = \mu B$ - Zeeman energy. The energy of the states with spin up and down differ by $2\omega_B$. Therefore, the ground state of a non-interacting system is described by two Fermi spheres with radii given by $\frac{p^2_{F\uparrow(\downarrow)}}{2m} = \epsilon_F \pm \omega_B$.

a) Find Green's functions $G_\uparrow(\epsilon, p)$ and $G_\downarrow(\epsilon, p)$ in a mean field (Hartree-Fock) approximation. For this purpose calculate the proper self-energy part in the lowest order in interaction g considering two contributions: direct and exchange shown in Figure.

b) Calculate the densities of particles $n_\uparrow$ and $n_\downarrow$ in the ground state.

c) Consider the polarization $n_\downarrow - n_\uparrow$ in the weak field $\mu B \ll \epsilon_F$ and show that the paramagnetic (Pauli) susceptibility is given by

$$\chi = \frac{\chi_0}{1 - g\nu_0}, \tag{2}$$

where $\chi_0 = \mu^2 \nu$ is a susceptibility of a non-interacting gas. The denominator $1 - g\nu_0$ describes so called exchange enhancement of the susceptibility occurring due to interaction.

Hint: See problem 1 of the HW 4.

Exercise 3: Hartree-Fock for long and short range potentials

4.1. A uniform spin-s Fermi system has a spin-independent interaction potential $V(x) = V_0 \frac{e^{-x/a}}{x}$.

(a) Evaluate the proper self-energy in the Hartree-Fock approximation. Hence find the excitation spectrum ϵ_k and the Fermi energy $\epsilon_F = \mu$.

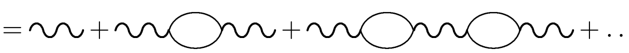
(b) Show that the exchange contribution to ϵ_F is negligible for a long-range interaction ($k_F a \gg 1$) but that the direct and exchange terms are comparable for a short-range interaction ($k_F a \ll 1$).

(c) In this approximation prove that the effective mass m^* is determined solely by *the exchange contribution*. Compute m^*, and discuss the limiting cases $k_F a \gg 1$ and $k_F a \ll 1$.

(d) What is the relation between the limit $a \to \infty$ of this model and the electron gas in a uniform positive background?

Exercise 4: Plasma waves

Let us consider diagrams shown in (3) at the finite frequency ω and momentum k. It is said that the sum shown in (3) describes the effect of *dynamic screening* of bare interaction V_k shown by wavy line. The dispersion law of *collective excitations* $\omega(k)$ is defined by poles of the screened interaction.

$V_{eff}(k, \omega) =$  (3)

a) Find the polarization operator $\Pi(\mathbf{k}, \omega)$ for $k \ll k_F$, $\omega \ll \epsilon_F$. Assuming that the bare interaction is Coulomb $V_k = 4\pi e^2/k^2$ sum the series and find the screened interaction $V_{eff}(k, \omega)$. Show that in this approximation the dispersion of plasma waves is given by

$$\frac{\omega}{2v_F k} \ln\left(\frac{\omega + v_F k}{\omega - v_F k}\right) - 1 = \frac{k^2}{4\pi e^2 \nu}, \tag{4}$$

where ν is a density of states.

b) Find $\omega(k)$ for small and large k.

Hint: See problem 7 of the HW 4.

Exercise 5: Debye screening

Electrons in metal screen any external electrostatic potential. This effect can be studied using the diagram series shown in (3). Find the sum of this series in the static limit $k \gg \omega/v_F$, also taking into account that $k \ll k_F$. Obtain the Debye formula for the screened Coulomb interaction:

$$\Phi(r) = \frac{e^{-\kappa r}}{r}, \qquad \text{where } \kappa^2 = 4\pi e^2 \nu. \tag{5}$$

Do you know how to obtain this formula classically?