Physics 556: Solid State Physics II

Homework 6

Read: FW 13-16, 4-5

“FW” refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1: Density-density response function in \(d = 1\)
Consider one-dimensional Fermi gas in an external field

\[
H_{\text{int}} = -\int dx \, e\phi(x,t)\hat{n}(x,t).
\]

The linear response of the density to the field \(e\phi(x,t)\) can be written in Fourier representation as

\[
\langle \hat{n}(k,\omega) \rangle = Q(k,\omega)e\phi(k,\omega).
\]

Find the response function \(Q(k,\omega)\) for small \(|k| \ll k_F\) and \(|\omega| \ll \epsilon_F/\hbar\).

Exercise 2: Plasmon in two dimensions

Derive the dispersion of plasma oscillations \(\Omega(q)\) for small \(q\) in two dimensions in RPA approximation. Assume a jellium model of a 2d electron gas with particle density \(n\), particle mass \(m\) and 3d Coulomb interaction \(V_0(x) = e^2/x\).

Exercise 3: Two-particle correlations (FW 5.10)

Two-particle correlations in the ground state can be characterized by the function

\[
g(x,y) = \langle \Psi_0 | \hat{\psi}_\alpha^\dagger(x) \hat{\psi}_\beta^\dagger(y) \hat{\psi}_\beta(x) \hat{\psi}_\alpha(x) | \Psi_0 \rangle - \langle \Psi_0 | \hat{\psi}_\alpha^\dagger(x) \hat{\psi}_\alpha(x) | \Psi_0 \rangle \langle \Psi_0 | \hat{\psi}_\beta^\dagger(y) \hat{\psi}_\beta(y) | \Psi_0 \rangle - \langle \Psi_0 | \hat{\psi}_\alpha^\dagger(x) \hat{\psi}_\alpha(x) | \Psi_0 \rangle \langle \Psi_0 | \hat{\psi}_\beta^\dagger(y) \hat{\psi}_\beta(x) | \Psi_0 \rangle.
\]

Show that for a non-interacting spin-\(\frac{1}{2}\) Fermi gas

\[
g^{(0)}(|x-y|) = -\frac{1}{2\rho_0^2} \left[ \frac{3j_1(k_F|x-y|)}{k_F|x-y|} \right]^2,
\]

where \(j_1(x)\) is a spherical Bessel function.

Exercise 4: Ultrarelativistic gas (FW 2.2)

Given the energy spectrum \(\epsilon_p = \sqrt{(pc)^2 + m_0^2c^4} \rightarrow pc\ (p \rightarrow \infty)\), prove that an ultrarelativistic ideal gas satisfies the equation of state \(PV = E/3\) where \(E\) is the total energy.

Exercise 5: BEC in 2d (FW 2.3)

Show that there is no Bose-Einstein condensation at any finite temperature for a two-dimensional ideal Bose gas with spectrum \(\epsilon_p^0 = \frac{p^2}{2m}\). What happens if one considers the ideal gas but with a modified spectrum \(\epsilon_p = \sqrt{\epsilon_p^0 (\epsilon_p^0 + 2mc^2)}\)? This modification can occur as a result of interactions between bosons.
Exercise 6: Paramagnetic spin susceptibility (FW 2.6)

Prove that the paramagnetic spin susceptibility of a free Fermi gas of spin-$\frac{1}{2}$ particles at $T = 0$ is given by $\chi(T = 0) = \frac{3}{2} \frac{\mu_0^2}{\epsilon_F} \frac{N}{V}$ where $\mu_0$ is the magnetic moment of one of the particles. Derive the corresponding high-temperature result $\chi(T \to \infty) = \frac{\mu_0^2}{k_B T} \frac{N}{V}$.