Physics 556: Solid State Physics II

*Homework 7

Read: **FW** 32, **AGD** Ch. 7

"**FW**" refers to Fetter, Walecka book.
Problems with stars are not for credit and will NOT be graded.

Exercise 1: Matsubara susceptibility and Curie's law

Consider a single free spin $s = 1/2$ at temperature T. The Hamiltonian of such system is zero. Let us define the Matsubara susceptibility with respect to an external weak magnetic field as

$$\chi_M(\omega_n) = \mu^2 \int_0^\beta d\tau \, e^{i\omega_n \tau} \langle T_\tau \left[\hat{s}_z(\tau) \hat{s}_z(0) \right] \rangle,$$

where μ is a magnetic moment of the spin. Notice that in the definition of the Matsubara susceptibility we have a chronologically ordered product of operators, not the retarded commutator as for a conventional Kubo susceptibility.

The theorem about analytical continuation states that one can obtain the Kubo susceptibility $\chi(\omega)$ by analytic continuation from the values of Matsubara susceptibility on the discrete set of frequencies on the positive part of imaginary axis $\omega = i\omega_n$, $n > 0$ to the real values of ω.

a) Calculate $\chi_M(\omega_n)$ for a single free spin at temperature T.

b) Continue the result analytically to the whole complex plane of ω and obtain Kubo susceptibility $\chi_K(\omega)$ so that $\chi_K(i\omega_n) = \chi_M(\omega_n)$ for $n > 0$.

c) In the static limit $\chi_K(\omega \to 0)$ obtain Curie's law.

Exercise 2: Superconducting instability

The pair susceptibility is defined as a linear response susceptibility of $\langle \psi^\dagger_\uparrow(x) \psi^\dagger_\downarrow(x) \rangle$ to the pairing field V, i.e., to the perturbation

$$H_{pair} = \int d^3x \, V(x,t) \left[\psi^\dagger_\uparrow(x) \psi^\dagger_\downarrow(x) + h.c. \right].$$

a) Argue that the diagram shown in Figure 1 (notice the direction of arrows) is the first diagram for Matsubara pairing susceptibility.
b) Calculate this diagram in the simplest case: $\chi^{(0)}_{pair}(q = 0, \omega_n = 0)$ i.e., for the static case $q = 0$ and for the lowest harmonics $\omega_n = 0$. Cut off the divergence at high momenta by the maximal energy E (relative to the Fermi energy, i.e., $|\xi_k| < E$).

Hint: Use $\xi_k = \hbar^{-1}(\epsilon_k - \mu)$ and $\frac{d^3k}{(2\pi)^3} \to \nu_F \, d\xi_k$ for isotropic integrals dominated by the vicinity of the Fermi surface.

Figure 1: Pair susceptibility to Ex. 2

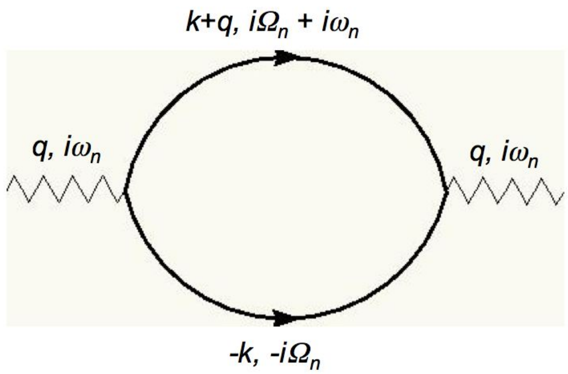
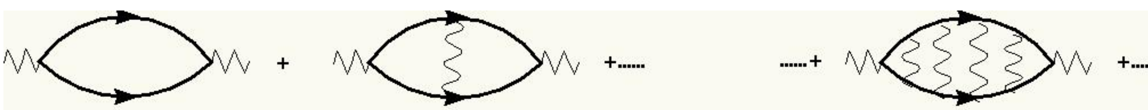


Figure 2: Pair susceptibility to Ex. 2



c) As the diagram is divergent we will sum the leading divergencies shown in the Figure 2 and obtain an RPA formula

$$\chi_{pair}(q,\omega_n) = \frac{\chi^{(0)}_{pair}(q,\omega)}{1 + U\chi^{(0)}_{pair}(q,\omega)},$$

where it is assumed that the interparticle interaction is a local one (essentially constant U in Fourrier space).

Study the behavior of $\chi_{pair}(q = 0, \omega_n = 0)$ for the case of a very weak attraction between fermions $U < 0$ and U is small. Show that as one decreases the temperature there is always an instability (the divergence of χ_{pair}) at some finite temperature T_c. Find an explicit formula for T_c in terms of U and cutoff energy E.

This instability is called *Cooper instability* and signifies the phase transition into the superconducting state. T_c is the superconducting critical temperature.